On the Game-Theoretic Nature of Optimality in Active Hypothesis Testing

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Active Hypothesis Testing

- m bandit hypotheses, each with K arms.
  Unknown bandit $i \in [m]$ realized.

- A **fixed** period of possibly *adaptive* experimentation.
  Protocol: Pull arm $a_t \in [K]$ and observe iid reward $X_t \sim p^{(i)}{}_{a_t}$.

- After experimentation, a final hypothesis $\hat{\imath}(a^T, X^T) \in [m]$ is chosen.

- Algorithms evaluated based on *error-probability* $P_i(\hat{\imath} \neq i)$.

Motivation: Models adaptive *pure* exploration in wide-ranging applications like channel selection in communication, sensor placement, designing drug trials.

Problem Formulation

- Error probability decays $P_i(\hat{\imath} \neq i) \approx e^{-E(A,i)T}$ as function of budget T. $E(A,i)$ - error exponent for Algo A on hypothesis i.
- Minimax optimality - $min_i E(A,i)$ is the worst case error decay rate.

  Come up with A, so that $min_i E(A,i)$ is maximized.
- The best worst-case performance desired.

  Example of Robust optimization but can be conservative.
- Technicalities - a) Simple hypotheses b) finite common full support $\mathcal{X}$ for arms

Related Work.

- Many different communities - IT, OR, Bandits.
- Sequential cousin well-understood - Wald, Chernoff,

  Javidi-Nagshvar, Cappe-Garivier-Kauffmann, Russo-Qin, Degenne-Koolen.
- **H**ayashi (2008) - simple static strategy optimal for Binary case (m=2).
- **N**itinawarat et al. (2013) - bounds for m>2, show adaptivity required.
- **K**omiyama et al. (2022) - Characterized optimal exponent E^*, **but** complicated, computationally intractable.

Difficult Old Problem

N'13 - *"In general, we still do not know the structure of the optimal causal control, and characterizing the optimal error exponent for causal control is a hard problem even for K=3."*

K'22 - **Remark 5.** (Utility of the DOT algorithm) Although DOT (Algorithm 2) has an asymptotically optimal rate R^{go}_{∞}, it is difficult to calculate, or to even approximate, the optimal solution of Eq. (3) since it is not an optimization of a finite-dimensional vector but an optimization of function $\boldsymbol{r}^B$, which has a high input dimension proportional to B. In this sense, the DOT algorithm as well as Theorem 7 is purely theoretical thus far, and the existence of a computationally tractable and provably optimal algorithm is an important open question.

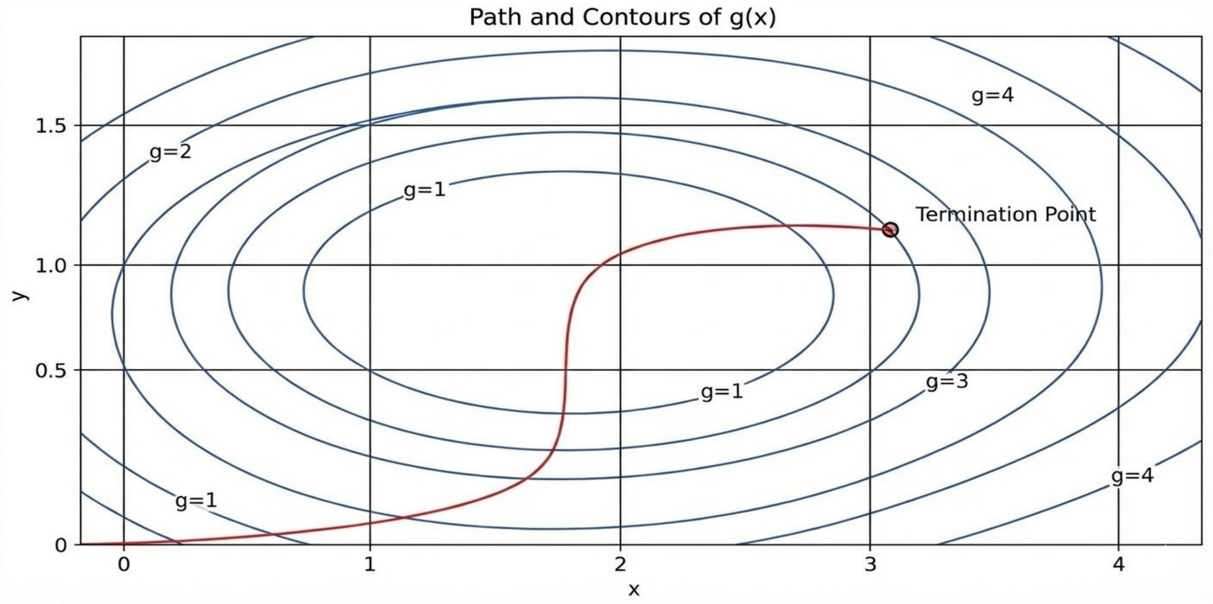
$\boldsymbol{E}^*$ as Value of Likelihood Game

- Game played in time interval $[0,1]$. At each $t \in [0,1]$, the ***agent*** plays an allocation $w_t \in \Delta_K$ and ***Nature*** plays $Q_t \in (\Delta_{\mathcal{X}})^K$.
- The dynamics evolves according to, for each $i \in [m]$:

$$\frac{dx_i(t)}{dt} = \sum_a w_t(a) D(Q_{a,t} \| p_a^i)$$

where the vector $x(t)$ is the state variable, $\mathrm{x}(0) = \mathbf{0}$ and encodes NLL.

- Both players vie for the terminal cost $g\big(x(1)\big) = x(1)_{(2)}$. Motivated by ML rule.



The PDE form

<u>Thm</u>: Given HJI-PDE with terminal condition:

$$\frac{\partial V}{\partial t} + \min_{Q}\max_{w}\left\{\sum_{i=1}^{m}\sum_{a\in[K]} w_t(a) D(Q_{t,a} \| p_a^i) \frac{\partial V}{\partial x_i}\right\} = 0$$

$$V(x,1) = g(x), \quad \forall x \in R^m,$$

then we have that: $E^* = V(0,0)$.

- $V(x,t)$ interpreted as Bellman-Isaac Value function.
- V uniquely solves the PDE weakly in the *Viscosity* sense. Consequence of the differential game representation and DPP.
- The minmax optimization is the Hamiltonian H. minmax =maxmin, hence order of play does not matter. We can give close form for H.
- We can leverage PDE methods to approximate and develop algos!

Consequences of PDE form

- Stability + Hopf/Lax gives improved bounds on E^* than **N'13.**

- Explicit Finite Difference (FD) scheme - accurately approximate E^* in contrast to **K'22**. Due to "Milder" *curse of dimensionality in* m rather than in B or T.

$$V_i^{n+1} = V_i^n - \Delta t \tilde{H}\left(\left(\frac{\Delta_j V_i^n}{\Delta h}\right)_{j\in[m]}\right)$$

- Convergence rate of $\sqrt{\Delta t}$ under **CFL** condition of $\Delta t = O(\Delta h/m)$. High accuracy for $m \leq 4$ but fails for higher m.

- Numerical disproof of a conjecture in **K'22** based on the FD scheme.

Deeper Look into the PDE

- $(x+\lambda 1)_{(2)} = (x)_{(2)} \Longrightarrow V(x+\lambda 1, t) = V(x,t) + \lambda$: Additive Homogenity of the solution.

- $x \preccurlyeq y$, then $(x)_{(2)} \leq (y)_{(2)} \Longrightarrow V(x,t) \leq V(y,t)$: Monotonicity of the solution.

- $(\lambda \cdot x)_{(2)} = \lambda \cdot (x)_{(2)} \Longrightarrow V(x,t) = (1-t)V\left(0, \frac{x}{1-t}\right)$: Scaling-in-time of the solution.

Implications

- Scaling condition $\Longrightarrow$ Geomteric problem:

$$V(x) = \langle \nabla V(x), x \rangle - H\big(\nabla V(x)\big)$$

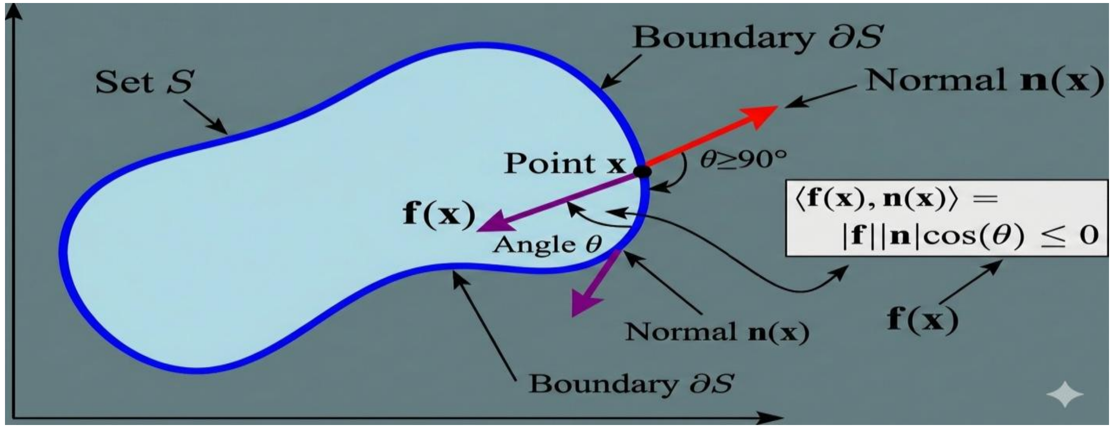
  with appropriate growth condition. A Clairaut PDE.

- Monotonicity + Additive Homogenity $\Longrightarrow$ $\nabla V \in \Delta_m$ Gradients in the simplex.

- Additive Homogenity $\Longrightarrow$ Only the zero-level set of V matters.

Invariance and Geometry

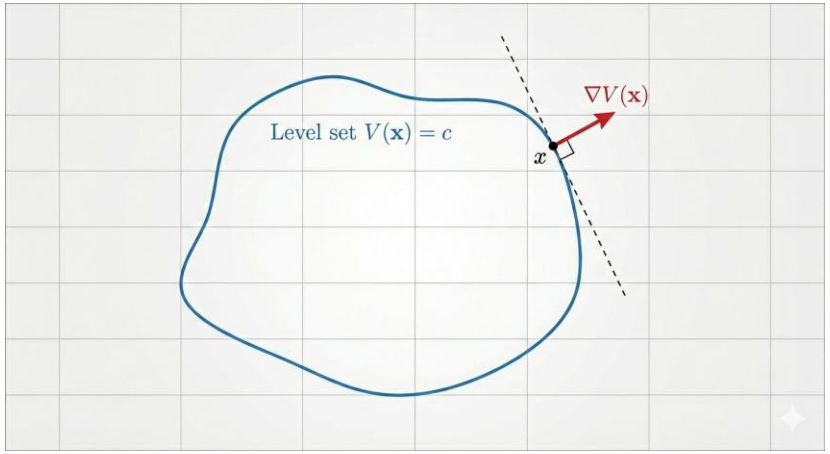
- Given dynamics $f(x)$, set S is invariant if dynamics never leaves, once it enters S. Sufficient condition:



- Dynamics of average:

$$\frac{d(\mathrm{x}(t)/t)}{dt} = \big(f\big(x(t)\big) - x(t)/t\big)/t$$

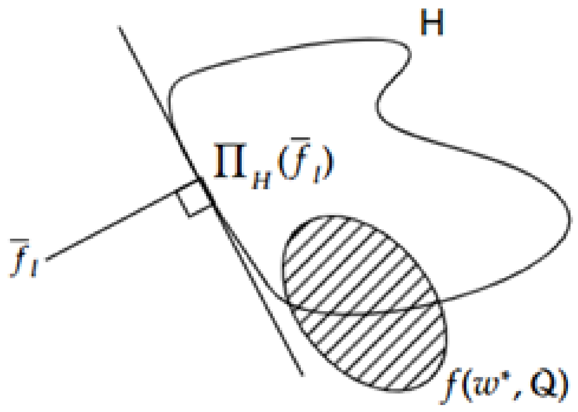
- What is the normal to a level set of a function?



Blackwell, Clairaut and Invariance

- Invariance/Approachability in games – Agent can approach S despite Nature's efforts.

- Blackwell: Given $x(t)$, some w, s. that, $x(t)$and $f(w,Q)$ on opp. halfspaces of the projection plane:



- Condition same as flow invariance! Moreover:

$$V(x) = \langle \nabla V(x), x \rangle - H\big(\nabla V(x)\big)$$

means the Clairaut zero sublevel set is approachable!

Summary

- AHT problems fundamental to exploration with budget constraints.
- Optimality as a differential game. Likelihood game between **A** and **N**.
- PDE methods for approximation and designing optimal algos. Curse of dimensionality.
- PDEs encode a flow invariance geometry, that can be used to design more efficient approachability algos.