Randomized Feasibility Methods for Constrained Optimization with Adaptive Step Sizes

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ICML 2026

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Consider the constrained optimization problem

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$$\text{subject to } x \in X \cap Y$$

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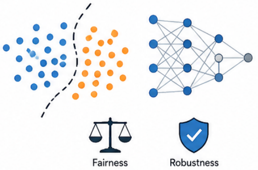
- set $Y \subseteq \mathbb{R}^n$ is closed, convex, and easy to project on (e.g., ball or box constraint)
- Set $X \subseteq \mathbb{R}^n$ is not easy to project on and is represented as

$$X := \cap_{i=1}^m \{x \in \mathbb{R}^n \mid g_i(x) \leq 0\}$$

- $g_i : \mathbb{R}^n \to \mathbb{R}$ for all $i = 1, \ldots, m$ are convex, and m can be a **large number**.

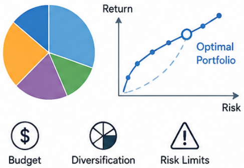
Motivating Applications

Fair & Robust Machine Learning



Optimize model performance while enforcing fairness and robustness constraints.

Portfolio Optimization

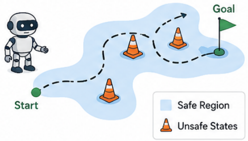


Maximize return subject to budget, risk, and regulatory constraints.

Energy Systems & Power Networks

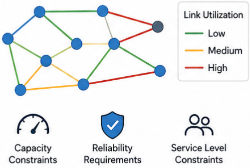
Optimize operation while satisfying capacity, reliability, and safety constraints.

Safe Reinforcement Learning



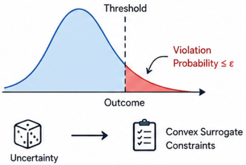
Learn policies that maximize reward while satisfying safety constraints.

Network Capacity & Reliability



Ensure reliable and efficient network operation under capacity constraints.

Chance-Constrained Optimization



Handle uncertainty via convex approximations of chance constraints.

Related Works and Motivation

Random Projections

- Sample a constraint and project onto it.
- Projection may be costly or unavailable.
- Can handle one constraint at a time.

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- Requires tuning or solving subproblems.
- Handles all constraints jointly.

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Randomized Feasibility Updates (Projection free)

- Subsample constraints for feasibility updates.
- Projection oracle not required.
- Scales for large/infinite constraints.

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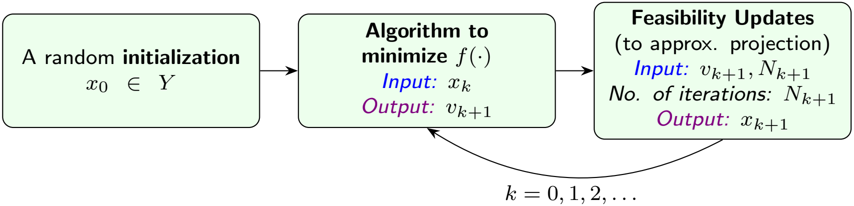
Motivation

Combine **randomized feasibility updates** with **adaptive step-size schemes** to:

- Achieve exponential convergence (up to a tolerance) for strongly convex objective function.
- Develop a fully parameter-free and line-search-free algorithm for convex optimization.

Our Algorithmic Overview



Optimization Algorithms

- Gradient Method
- Distance over Weighted (sub)Gradients[a] (DoWS)
- Tamed-Distance over Weighted (sub)Gradients (T-DoWS)

[a] Ahmed Khaled, Konstantin Mishchenko, and Chi Jin. "DoWG unleashed: An efficient universal parameter-free gradient descent method". In: *Advances in Neural Information Processing Systems* 36 (2023), pp. 6748–6769.

Randomized Feasibility Updates

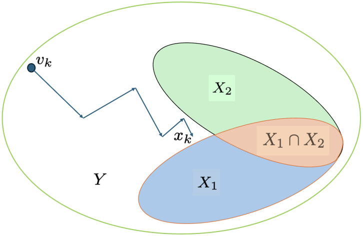
- Inner loop consists of N_k feasibility iterations.
- N_k is specified at every outer iteration $k \geq 1$.
- Sequential or parallel updates are possible.
- We consider sequential updates[a].

[a] Angelia Nedić and Ion Necoara. "Random minibatch subgradient algorithms for convex problems with functional constraints". In: *Applied Mathematics & Optimization* 80.3 (2019), pp. 801–833.

Randomized Feasibility Updates (Sequential Implementation)[1]

At iteration k of the Gradient Method

- **Require:** v_k, N_k constant step size $0 < \beta < 2$
- **Set:** $z_k^0 = v_k$
- **for** $i = 1, \ldots, N_k$
  - **Sample:** Choose a random index $\omega_k^i \in \{1, \ldots, m\}$
  - **Compute:** Subgradient d_k^i of $g^+_{\omega_k^i}(z)$ at $z = z_k^{i-1}$
  - **Update:**
    $$z_k^i = \Pi_Y \left[z_k^{i-1} - \beta \, \frac{g^+_{\omega_k^i}(z_k^{i-1})}{\|d_k^i\|^2} \, d_k^i \right]$$
- end for
- **Output** $x_k = z_k^{N_k}$



[1] Boris T. Polyak. "Minimization of unsmooth functionals". In: *USSR Computational Mathematics and Mathematical Physics* 9.3 (1969), pp. 14–29.

Gradient Method for Strongly Convex Objective Function

Assumptions: The function f is L-Lipschitz smooth and μ-strongly convex.

$$f(y) - f(x) \leq \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|y - x\|^2 \text{ for all } x, y \in Y.$$

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Gradient Method with Randomized Feasibility:

$$v_{k+1} = \Pi_Y[x_k - \alpha_k \nabla f(x_k)]$$
$$x_{k+1} = \texttt{Randomized Feasibility}(v_{k+1}, N_{k+1})$$

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Question: How to choose α_k?

Main Result: Strongly Convex Case

Theorem

- f is L-smooth and μ-strongly convex
- Sufficient sample selection:

$$\mathbb{E}\big[(1-q)^{N_k/2}\big] \leq \frac{\epsilon}{(B_0 + \|x^*\|)M_f}.$$

- Adaptive step size

$$\alpha_k = \min\left\{\frac{1}{2(L-\mu)}, \frac{1}{L}, \frac{\epsilon}{2\|\nabla f(x_k)\|^2}\right\}.$$

- Algorithm 1 produces an iterate $\bar{x}_k$ satisfying $\mathbb{E}[\,|f(\bar{x}_k) - f(x^*)|\,] \leq \epsilon$ after $k = O\big(\log \frac{1}{\epsilon}\big)$ iterations.

If $N_k = N$ for all $k \geq 1$, then sample selection to reach ϵ tolerance is

$$N \geq \frac{2\ln\left(\frac{(B_0+\|x^*\|)M_f}{\epsilon}\right)}{\ln\left(\frac{1}{1-q}\right)}.$$

Parameter-Free Step Sizes for Convex Nonsmooth Optimization

Convex Nonsmooth Setting

$$f(y) - f(x) \geq \langle s_f(x), y - x \rangle, \qquad s_f(x) \in \partial f(x).$$

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Algorithm Ingredients

- Adaptive radius estimate: $\bar{r}_k = \max\{\|x_k - x_0\|, \bar{r}_{k-1}\}$
- (Sub)gradient accumulation: $p_k = p_{k-1} + \bar{r}_k^2 \|s_f(x_k)\|^2$

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DoWS + Randomized Feasibility

$$\alpha_k = \frac{\bar{r}_k^2}{\sqrt{p_k}}$$

Assumption

- Y is bounded.

T-DoWS + Randomized Feasibility

$$\alpha_k = \begin{cases} \dfrac{\bar{r}_k^2}{2\sqrt{p_k}\ln\left(\frac{e p_k}{p_1}\right)} & p_0 = 0, \\ \dfrac{\bar{r}_k^2}{\sqrt{2p_k}\ln\left(\frac{e p_k}{p_0}\right)} & p_0 > 0. \end{cases}$$

Assumption

- Level set $\{y \in Y \mid f(y) \leq f^*\}$ is bounded

Convergence Rates of the Parameter-Free Methods

Let **RF** denotes Randomized Feasibility Algorithm.

Theorem

For any $T \geq 1$, define $\tau = \arg\min_{1 \leq k \leq T} \frac{\bar{r}_{k+1}^2}{\sum_{i=1}^{k} \bar{r}_i^2}$ and $\bar{x}_\tau = \frac{\sum_{k=1}^{\tau} \bar{r}_k^2 x_k}{\sum_{i=1}^{\tau} \bar{r}_i^2}$. Then

$$\mathbb{E}\left[|f(\bar{x}_\tau) - f^*|\right] \leq \max\{A_1(T), \min\{A_2(T), A_3(T)\}\}.$$

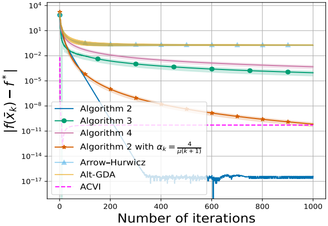
Term	**DoWS + RF**	**T-DoWS + RF**
$A_1(T)$	$O(1/\sqrt{T})$	$O(\ln(T)/\sqrt{T})$
$A_2(T)$	$O\left(\max_{1\leq k\leq T} \mathbb{E}\left[(1-q)^{N_k/2}\right]\right)$	$O\left(\max_{1\leq k\leq T} \mathbb{E}\left[(1-q)^{N_k/2}\right]\right)$
$A_3(T)$	$O\left(\frac{\sum_{k=1}^{T} \mathbb{E}\left[(1-q)^{N_k/2}\right]}{T}\right)$	$O\left(\frac{\sum_{k=1}^{T} \mathbb{E}\left[(1-q)^{N_k/2}\right]}{T}\right)$

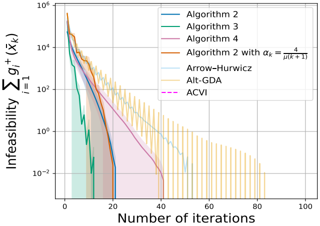
- **E.g.:** If $N_k = \lceil k^{\frac{1}{p}} \rceil$ and $p > 0$, then $A_3(T) = O(1/T)$ for both algorithms.

Simulation (Quadratically Constrained Quadratic Programming)

$$\min_{x\in[-10,10]^{10}} \langle x, Ax\rangle + \langle b, x\rangle$$
$$\text{s.t. } \langle x, C_i x\rangle + \langle u_i, x\rangle - e_i \le 0 \quad \text{for } i = 1,\dots,1000.$$

Strongly convex objective function with known f^*:

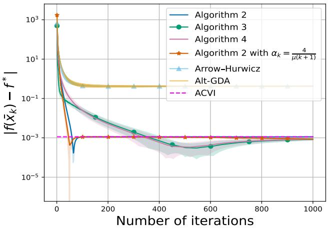


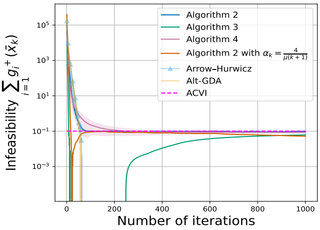


Algorithm 2: Gradient Method + Randomized Feasibility
Algorithm 3: DoWS + Randomized Feasibility
Algorithm 4: T-DoWS + Randomized Feasibility

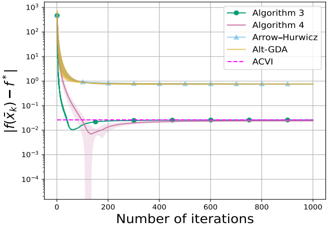
Simulation (Quadratically Constrained Quadratic Programming (QCQP))

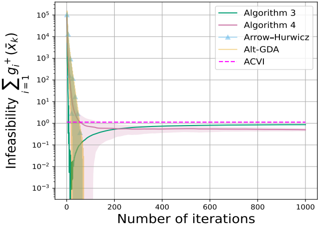
Strongly convex objective function with unknown f^*:



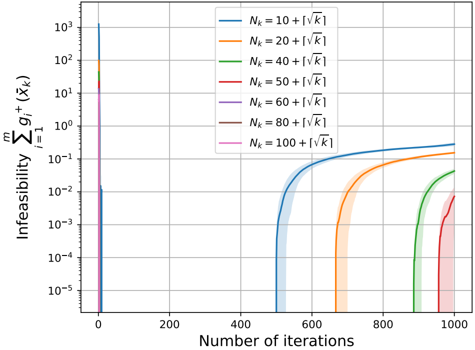


Convex objective function with unknown f^*:





QCQP: Effect of Feasibility Sample Size N_k



Setup

- Convex objective
- DoWS + Randomized Feasibility

Observation

- Larger N_k yields faster infeasibility reduction.
- The solution lies near the constraint boundary.
- Small N_k may drive infeasibility close to zero initially, but violations can reappear later.
- Too few sampled constraints may miss active constraints.

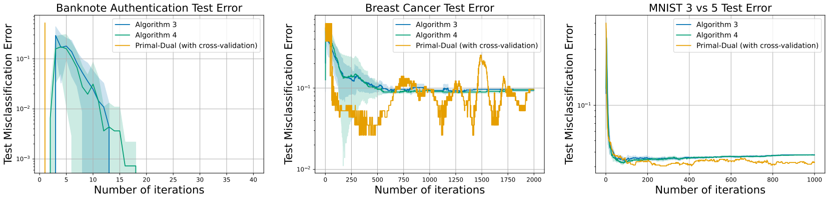
Takeaway

Increasing N_k improves feasibility accuracy at the cost of additional computation.

Simulation (Support Vector Machine)

$$\min_{w,b,\xi} \quad \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{m}\xi_i$$

$$\text{subject to} \quad 1 - \xi_i - y_i(w^\top z_i + b) \leq 0\ , \quad \xi_i \geq 0, \quad i = 1, \ldots, m.$$



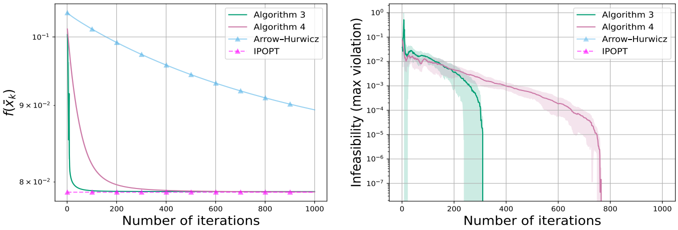
Algorithm 3: DoWS + Randomized Feasibility
Algorithm 4: T-DoWS + Randomized Feasibility

Simulation (Logistic Regression with Group Fairness Constraints)

$$\min_{x\in\mathbb{R}^n} \quad f(x) := \frac{1}{\widehat{N}} \sum_{j=1}^{\widehat{N}} \ln(1+\exp(-y_j \langle a_j, x\rangle))$$

$$\text{s.t.} \quad x \in \{x \in \mathbb{R}^n \mid \ell \le x \le u\} \bigcap \left\{ \bigcap_{i=1}^{m} \left\{ x \in \mathbb{R}^n \;\middle|\; \left|c_i^\top x\right| \le \tau_i \right\} \right\}$$

- See the manuscript for construction details



(a) Objective function value

(b) Maximum Infeasibility violation

Algorithm 3: DoWS + Randomized Feasibility
Algorithm 4: T-DoWS + Randomized Feasibility

More results and insights can be found in our work.

Check it out below!

Thank You!

Scan the QR code to read the full paper on arXiv.

Acknowledgements: This work is partially supported by the NSF award CIF 2134256.