Hyperbolic Neural Population Geometry Benefits Computation

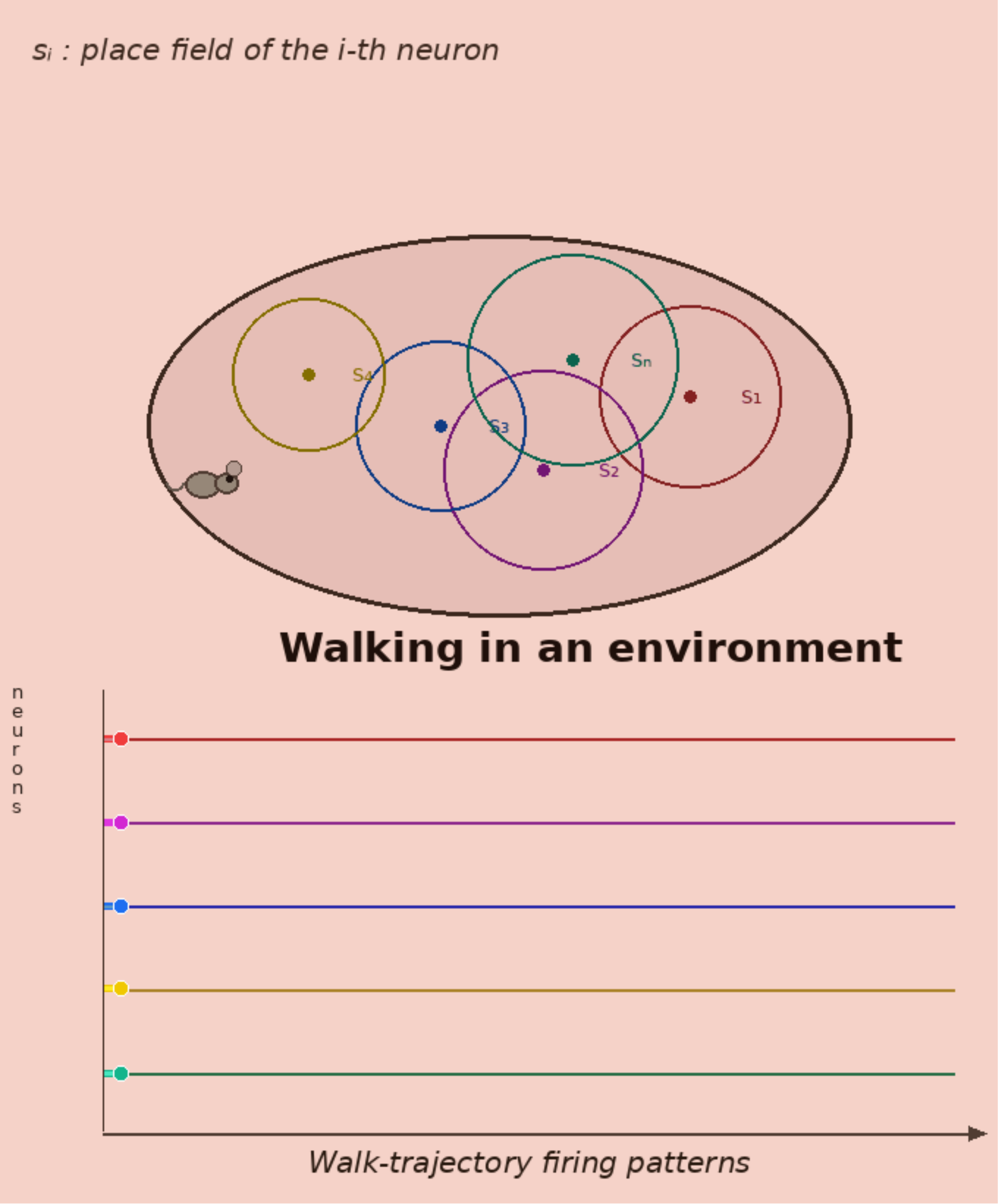
Dennis Wu[13], Yi-Chun Hung[1], Braden Yuille[1], James E. Fitzgerald[*12], Han Liu[*13]

[1]Northwestern University, [2]National Institute for Theory and Mathematics in Biology,
[3]Center of Foundation Models and Generative AI

Background and Motivation

Hippocampal Place Cells

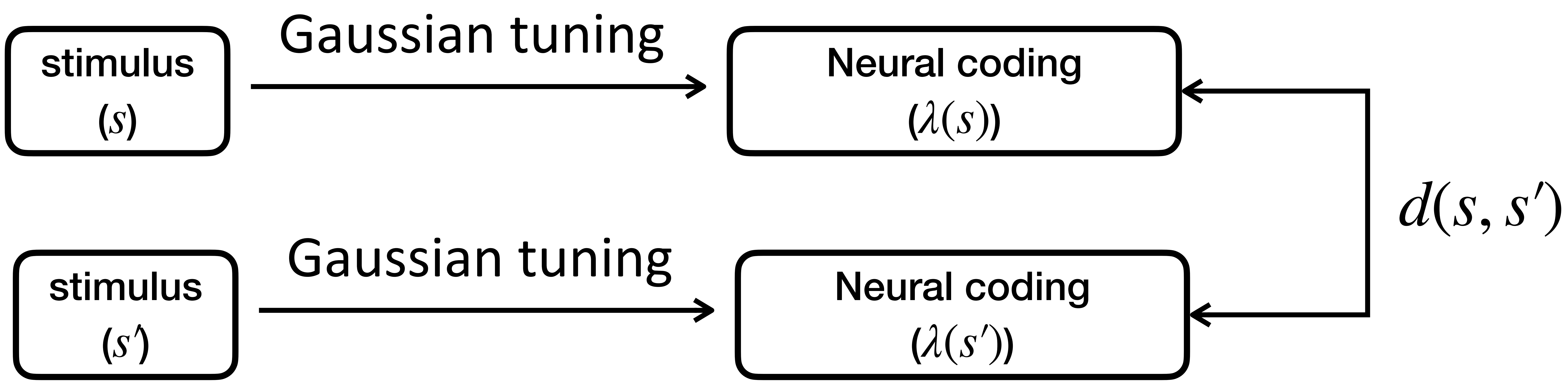
- Neurons fire selectively at spatial locations
- Their firing rate patterns form as high-dimensional embeddings of spatial information
- How do we detect the underlying geometry of this embedding space?



Research Question

- What geometry do hippocampal place cells actually encode?
- How can we exploit it for better memory models?

Setup



Gaussian tuning curves

$$\lambda_i(s) := \exp\left(-\frac{\|s - s_i\|_2^2}{2\sigma_i^2}\right), \quad \sigma_i \sim \text{Exp}(\beta)$$

Place field sizes follow exponential distribution [1]

Population codes

$$\lambda(s) = [\lambda_1(s), \cdots, \lambda_N(s)] \in \mathbb{R}^N$$

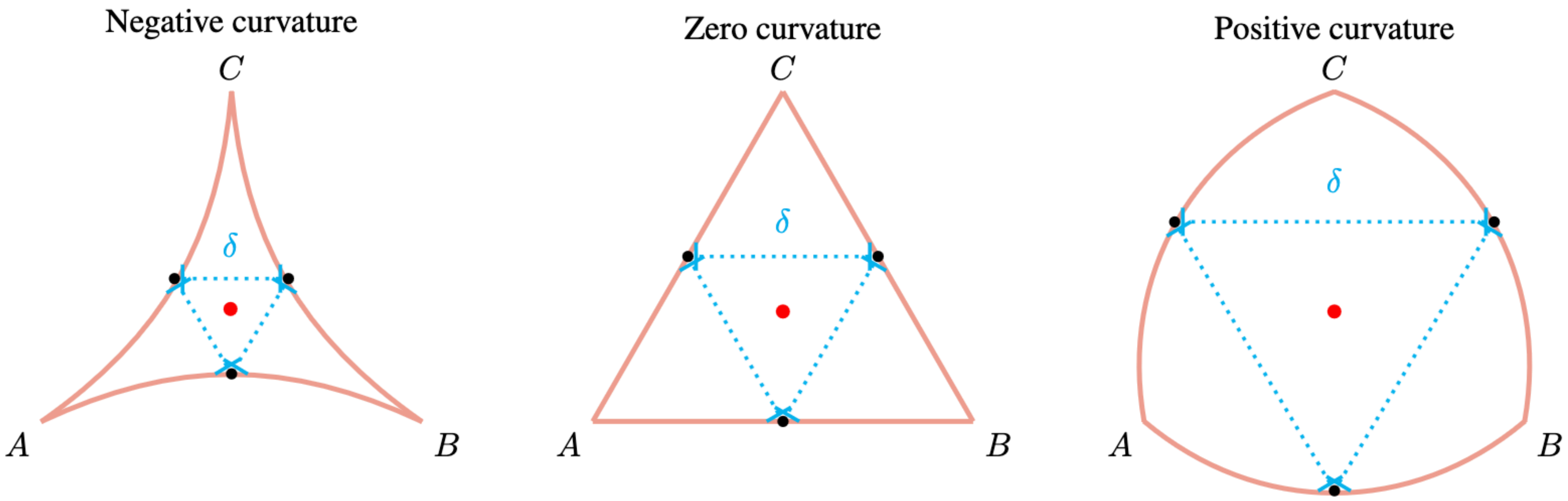
Distance function between population codes

$$d(s_a, s_b) := -\ln\left(\langle \lambda(s_a), \lambda(s_b) \rangle\right) + C$$

[1]: Zhang, H., Rich, P.D., Lee, A.K. *et al.* Hippocampal spatial representations exhibit a hyperbolic geometry that expands with experience. *Nat Neurosci* (2023).

Hyperbolic Structure

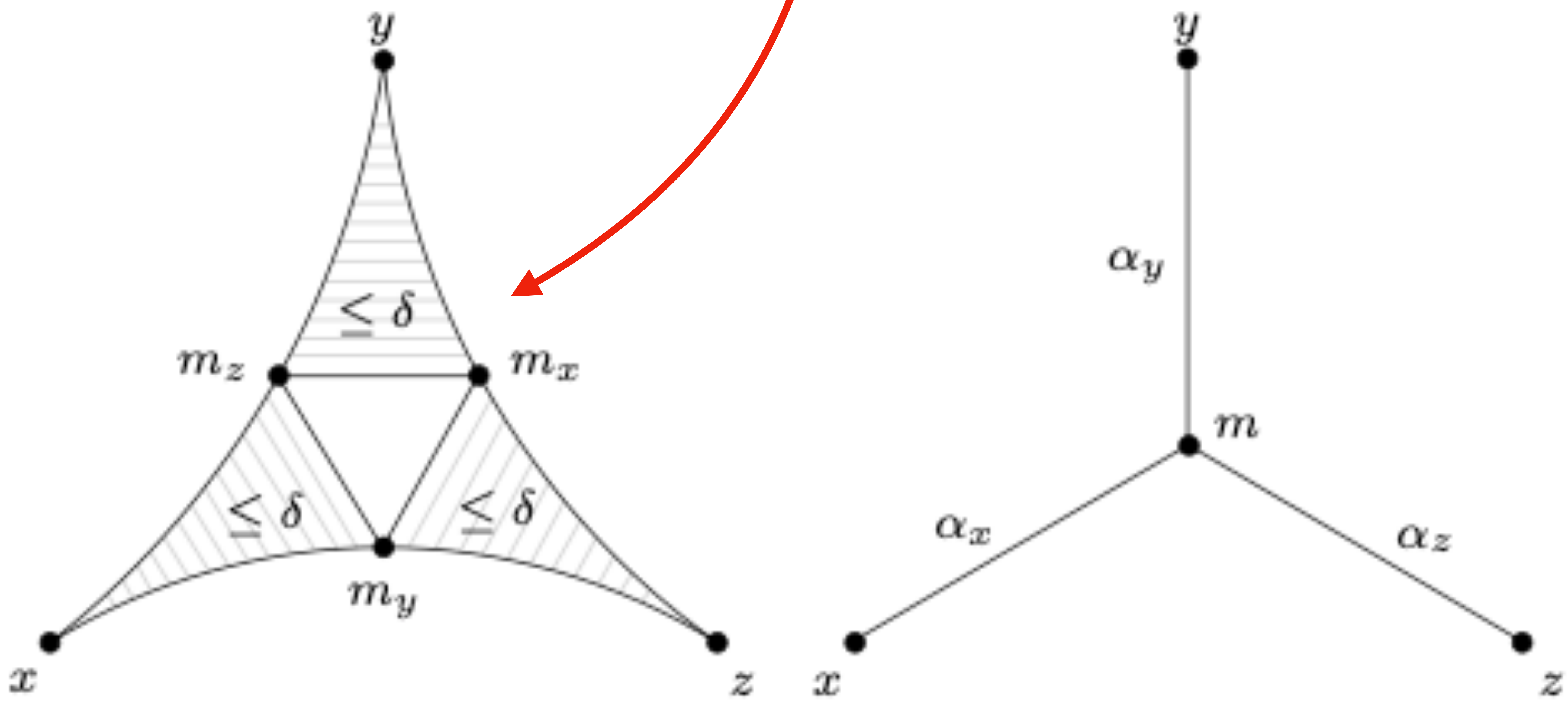
Gromov's notion of hyperbolic metric space: δ-thin triangles



We can detect geometric structures via triangles!!

Hyperbolic Structure

In hyperbolic space, $\delta = \ln(\sqrt{2} + 1)$ regardless of the size of the triangle



If δ is a constant independent to the size of the triangle, the underlying geometry is hyperbolic!!

Result

Theorem 4.4. *Let s_a, s_b, s_c be any three stimuli each independently sampled uniformly from the environment $[0, L]^D$. Let place field sizes be exponentially distributed, i.e., $\beta \sim Exp(\beta)$. For any $\eta > 0$ and $N = \mathcal{O}((L/\beta)^D)$, there exists a constant $\delta(\beta)$ such that*

$$Pr[\Delta(\lambda(s_a), \lambda(s_b), \lambda(s_c)) \text{ is } \delta\text{-thin}] > 1 - \eta.$$

Furthermore, δ is non-trivial since $\lim_{L \to \infty} \frac{\delta}{L} = 0$.

Place cells encode spatial information as hyperbolic embedding!!!

Hyperbolic associative memory networks

The Karcher-flow Model

- New hyperbolic associative memory network
    - Retrieval dynamics minimizes squared-geodesic loss defined in hyperbolic space
- Retrieval dynamics derived via the Karcher-flow algorithm

Hyperbolic associative memory networks

Theorem 4.8 (Informal). *Let $\kappa < 0$ to be the curvature and $\alpha = \sqrt{|\kappa|}$. Let $r_{\max}, r_{\min}$ be the largest and smallest norm of memory patterns. Let d be the dimensions of the memory patterns. Assume*

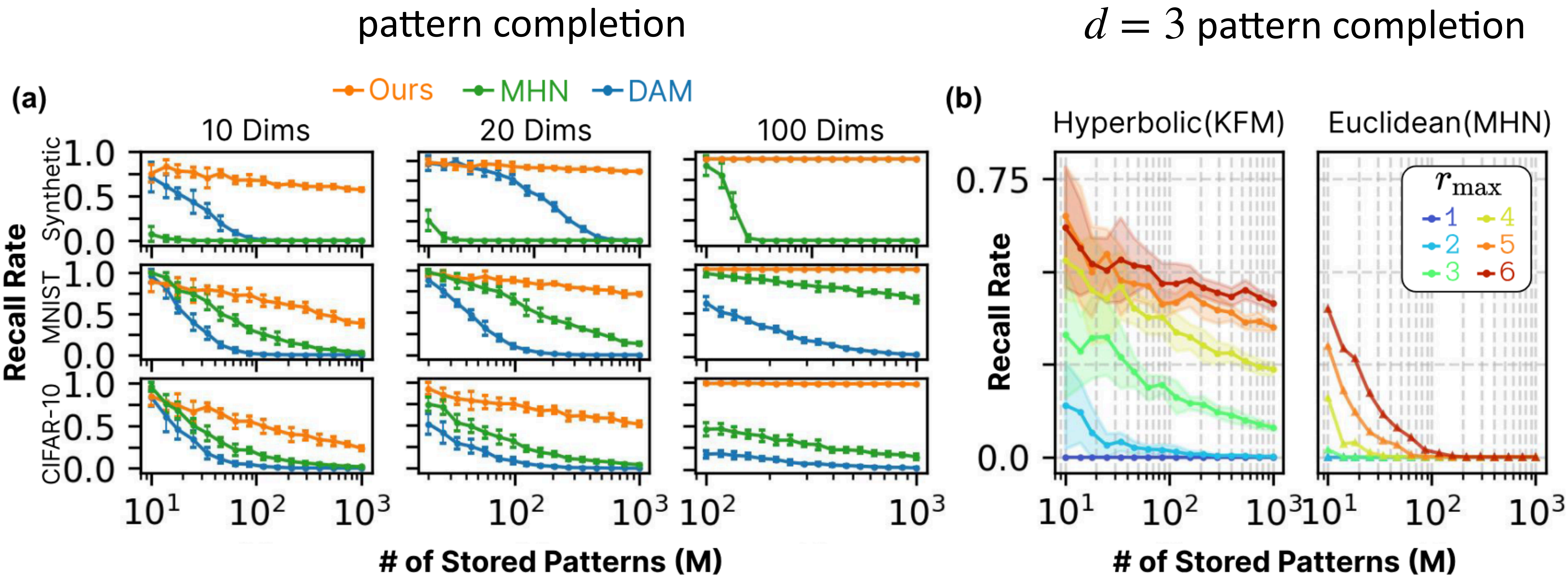
$$r_{\max} - r_{\min} = o\left(\frac{d}{\alpha r_{\min}^2}\right)$$

hold. Then the maximum number of memory pattern stored by our hyperbolic associative memory network M satisfies

$$\log M = \Theta\left(\frac{d}{|\kappa|} \frac{e^{2\alpha r_{\min}}}{r_{\min}^2}\right).$$

Memory capacity is exponential in pattern dimension, and double exponential in memory pattern radius!!!

Pattern Completion Simulation



Takeaways

- We prove that the hippocampal place cells form hyperbolic internal maps for spatial information
- We provide a novel mathematical framework to study the geometry of neural population codes
- We propose a new associative memory model operates in the hyperbolic space with significant capacity