Quantifying the Noise Sensitivity of the Wasserstein Metric for Images

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Signed Wasserstein discrepancies preserve image geometry under heavy pixel-wise noise
W_2 error grows like $\sqrt{\sigma}$, not like the linear Euclidean rate

Summary

We study how Wasserstein metrics for images behave under pixel-wise additive noise, treating images as discrete measures on a fixed grid. We prove bounds under on the expectation under zero-mean Gaussian pixel noise and show, empirically and theoretically, that Wasserstein discrepancies can be more robust to noise than the Euclidean distance.

Problem Setting

An image is a real-valued signal on an $n \times n$ grid G_n with $m = n^2$ pixels.
Additive noise is added to two clean images $\mu_\varepsilon = \mu + \varepsilon_\mu, \qquad \nu_\varepsilon = \nu + \varepsilon_\nu$.

Signed Wasserstein Discrepancy

Additive Gaussian noise introduces negative pixel values. Using the Jordan decompositions: $\mu = \mu_+ - \mu_-, \qquad \nu = \nu_+ - \nu_-$.

Define the signed discrepancy [1] :

$$W_p^{\pm}(\mu,\nu) := W_p(\mu_+ + \nu_-, \nu_+ + \mu_-).$$

This way, signed, noise-corrupted images can be compared using the standard Wasserstein metric between two non-negative measures.

Zero-Sum Gaussian Noise Model

We consider an almost-i.i.d. Gaussian noise vector $\mathrm{N} \sim \mathcal{N}(0,\Sigma)$ where:

$$\Sigma_{ij} = \begin{cases} \sigma^2, & i = j \\ -\dfrac{\sigma^2}{m-1}, & i \neq j \end{cases}.$$

This gives marginal variance σ^2 and guarantees no added mass:

$$N_i \sim \mathcal{N}(0,\sigma^2), \qquad \mathrm{N}_1 + \cdots + \mathrm{N}_m = 0.$$

Single Image results

For zero-sum noise $\varepsilon_1, \varepsilon_2$, $p > 1$ and $\mathrm{C_p} = \frac{2^{\frac{1}{p}-1}}{\sqrt{\pi}}\Gamma(\frac{1}{2p}+\frac{1}{2})$:

$$\mathrm{C_p} \cdot \mathrm{n}^{\frac{2}{p}-1} \cdot \sigma^{1/p} \leq \mathbb{E}\left[\left(W_p^{\pm}(\mu+\varepsilon_\mu, \mu+\varepsilon_\nu)\right)\right] \leq \left(\frac{4\sqrt{2}}{\sqrt{\pi}} n\sigma\right)^{1/p}$$

Two image results

Similarly, for zero-sum noise $\varepsilon_\mu, \varepsilon_\nu$, $p > 1$ and two measures μ, ν:

$$\mathbb{E}[W_p^{\pm}(\mu+\varepsilon_\mu, \nu+\varepsilon_\nu)] \leq \left(\frac{\sqrt{2}}{2}\right)^{1-\frac{1}{p}} \cdot \mathrm{W_1}\,(\mu,\nu)^{1/p} + \frac{\sqrt{2}}{2}\left(\frac{4}{\sqrt{\pi}}\mathrm{n}log_2(n) + \frac{1}{\sqrt{\pi}}n\right)^{1/p} \sigma^{1/p}$$

W_1 Bound

For two clean images: $\mathbb{E}[W_1^{\pm}(\mu_\varepsilon,\nu_\varepsilon) - W_1^{\pm}(\mu,\nu)] \leq \frac{4nlog_2 n + n}{\sqrt{\pi}}\sigma + \frac{\sqrt{2}}{\mathrm{n}}$

The inter-image geometry is preserved under controlled additive noise.

Inter-Image Robustness

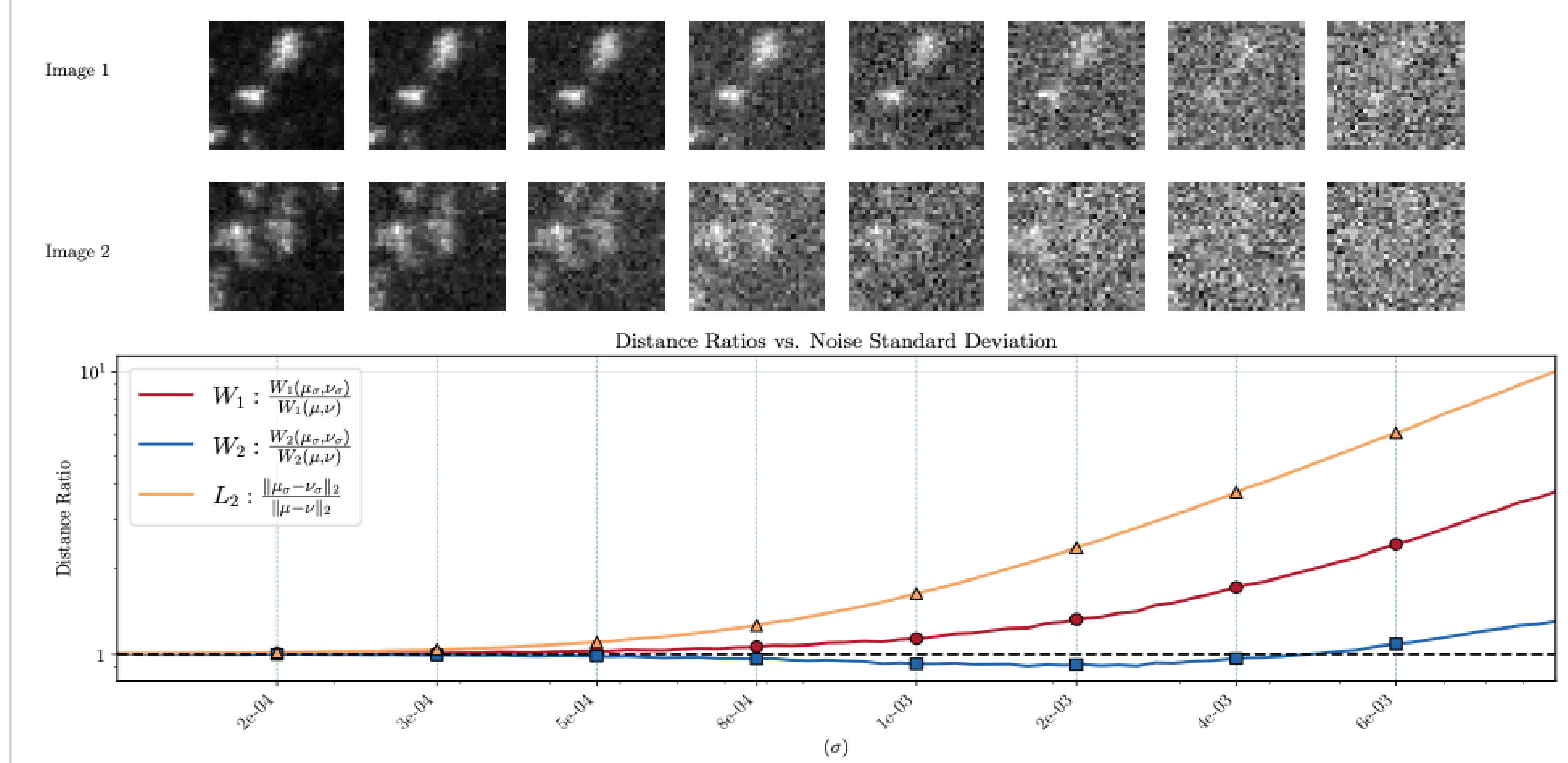
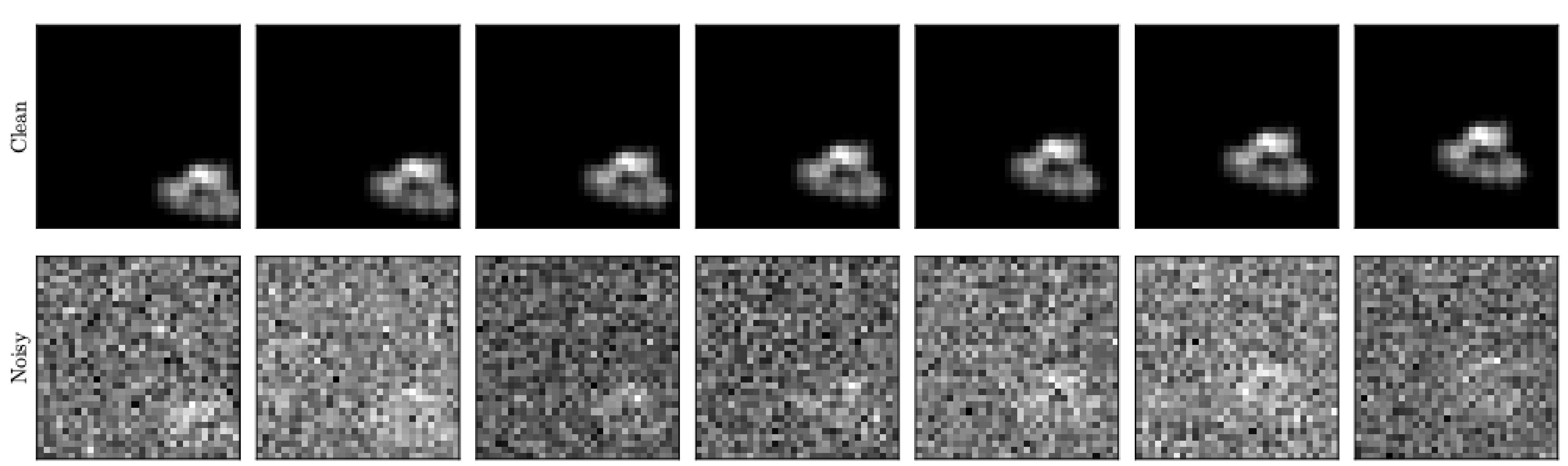
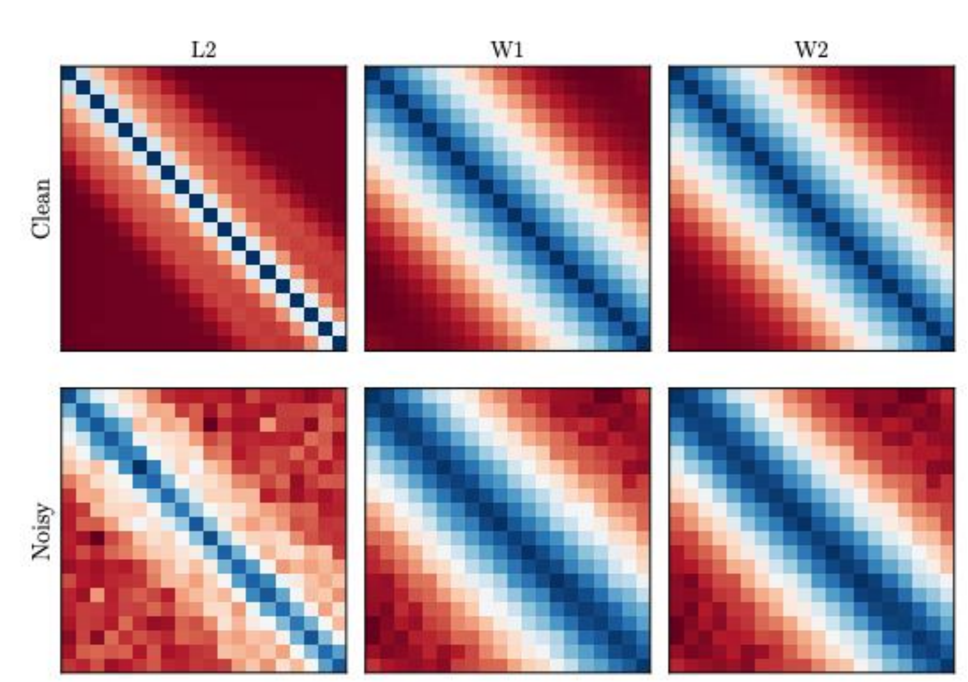


Fig. 1: *Distance ratios for a noisy image pair vs. noise level. A ratio near 1 means the noisy distance still reflects the clean-image distance.*

Cryo-EM Case Study





Figs 6–7: *Sample of clean vs. noisy E. coli Hsp90 projections (left) and pairwise distance matrices (right).*
Under heavy noise L_2 loses the clean diagonal structure, while W_1 and W_2 preserve the global geometry.

References

[1] Edoardo Mainini. *"A description of transport cost for signed measures"*, Journal of Mathematical Sciences (2012).

[2] Jonathan Weed, Francis Bach. *"Sharp asymptotic and finite-sample rates of convergence of empirical measures in Wasserstein distance"*. Bernoulli (2019).

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https://github.com/warik21/Quantifying-wasserstein-noise-sensitivity