Geometric Learning And Discovery

中國人民大學
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Planar Symmetric Pattern Generation

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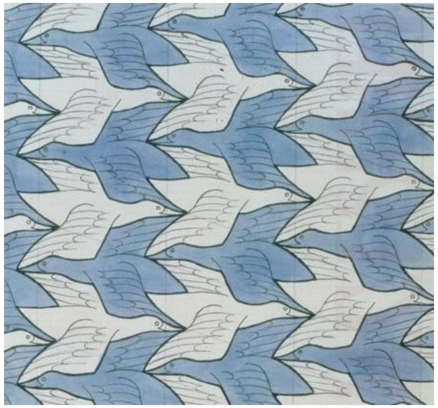
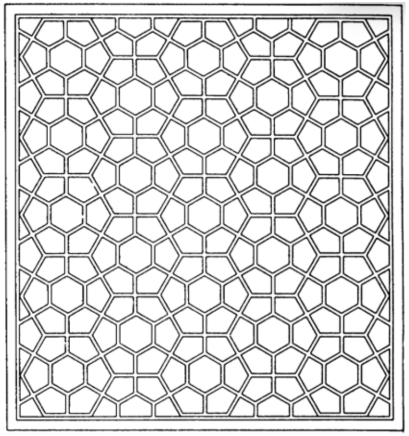
Symmetry in Manufacturing [1-4]

Periodic Drawing

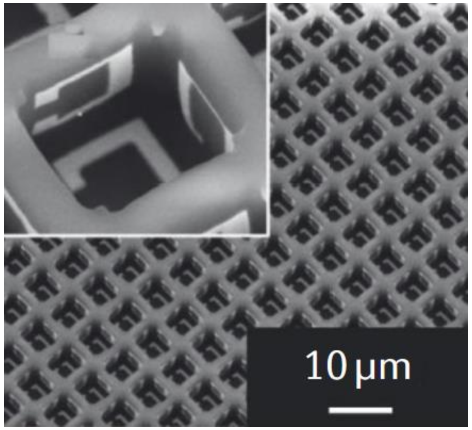


Chinese Paper-cutting

Chinese Window Lattice



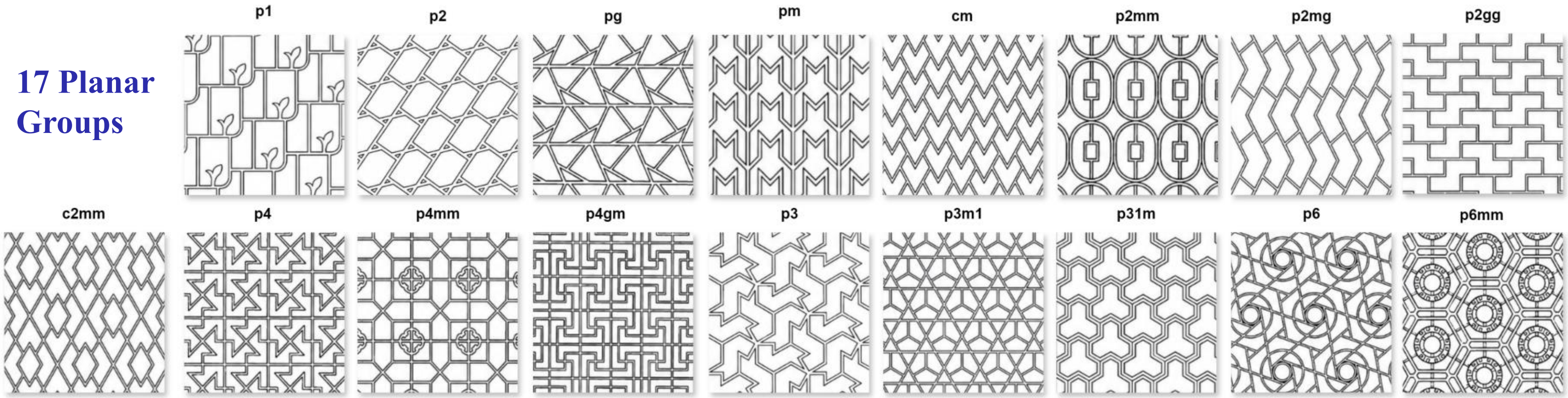
Metameterial



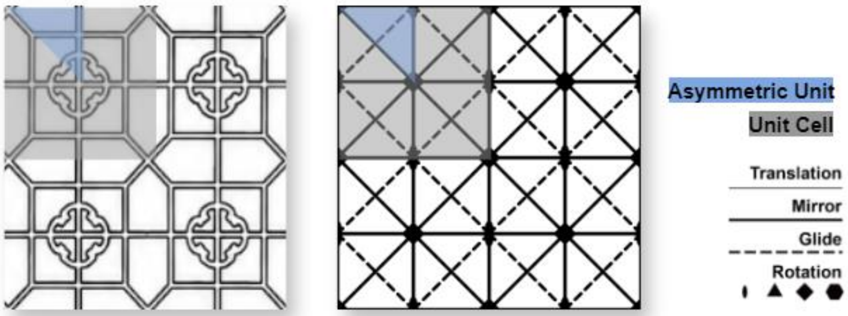
Planar Group Symmetry [5]

Planar group: a planar group is a discrete subgroup of planar Euclidean group that contains two linearly independent translations.

17 Planar Groups

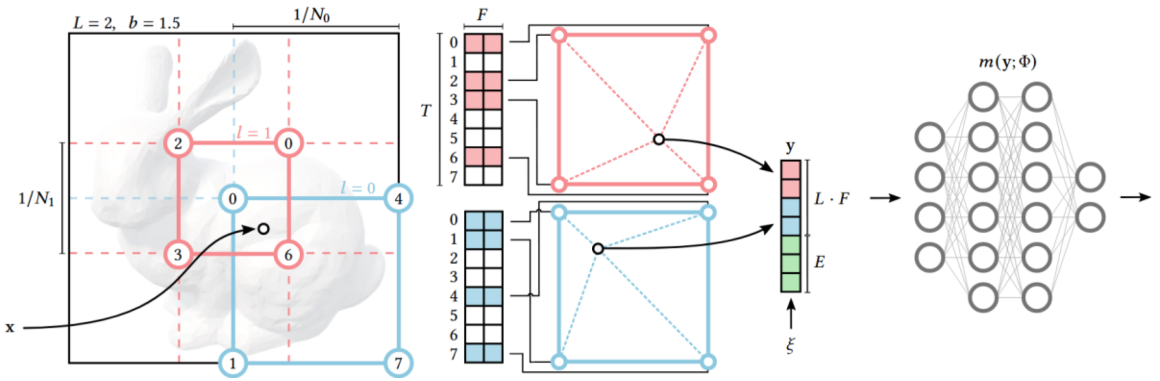


Asymmetric unit: the minimal, non-redundant subset of plane from which the structure is obtained through the symmetry operations of the group.

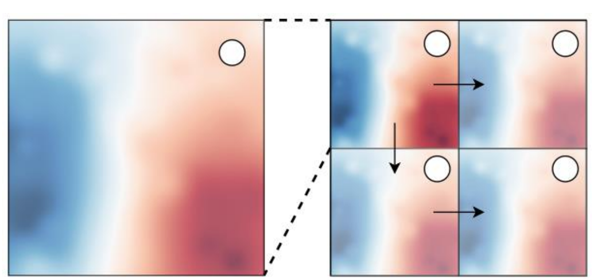


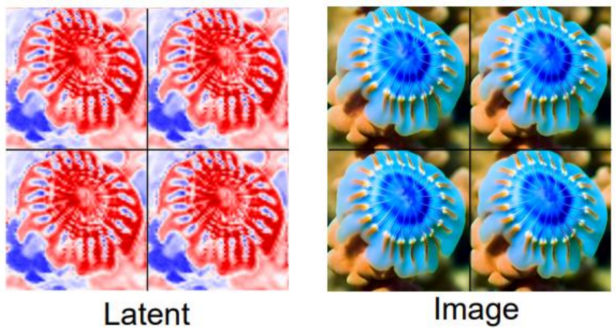
Symmetry Representation [6]

2D representation: Continuous representations play an important role in modeling 3D and 2D objects.



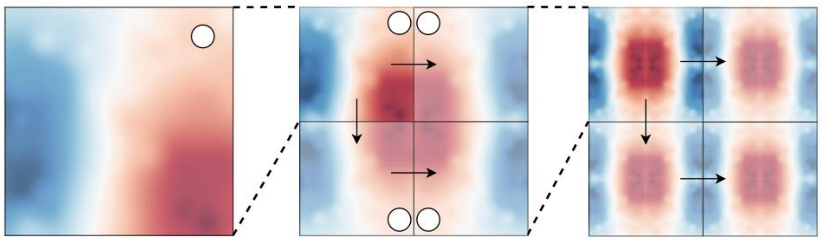
Naive symmetrization: Naive extension of continuous representation on asymmetric unit may introduce seams near boundaries.





Observation

Fact 1 (reflection preserves continuity): For affine reflection groups, the extension strategy preserves boundary continuity.



Fact 2 (decomposition for *p1* cases): Periodic function can be decomposed into a combination of high-symmetry coefficients and low-symmetry bases.

Planar Fourier Series

$$f(x,y)=\sum_{h,k\in\mathbb{Z}} c_{hk}e^{i(hx+ky)}$$

Euler Formula →

Basis Function

$$\begin{cases}\cos(hx)\cos(ky) & h\geq 0, k\geq 0\\ \cos(hx)\sin(ky) & h\geq 0, k>0\\ \sin(hx)\cos(ky) & h>0, k\geq 0\\ \sin(hx)\sin(ky) & h>0, k>0\end{cases}$$

Multiple-angle Formula

$$\begin{cases}\sin(nx)=U_{n-1}(\cos x)\sin x\\ \cos(nx)=T_n(\cos x)\end{cases}$$

↓

Expanded Function

$$\begin{cases}T_h(\cos x)T_k(\cos y) & h\geq 0, k\geq 0\\ T_h(\cos x)U_{k-1}(\cos y)\sin y & h\geq 0, k>0\\ U_{h-1}(\cos x)T_k(\cos y)\sin x & h>0, k\geq 0\\ U_{h-1}(\cos x)U_{k-1}(\cos y)\sin x\sin y & h>0, k>0\end{cases}$$

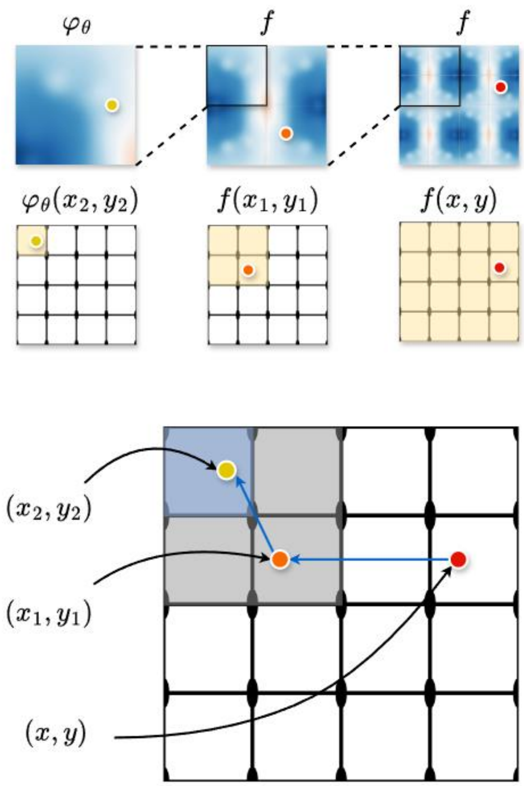
←

Decomposition

$$f(x,y)=\sum_{i=1}^{4}\underbrace{p_i(\cos x,\cos y)}_{\substack{\text{polynomial of cosine}\\ p2mm\text{ invariant}}}\ \underbrace{\eta_i(x,y)}_{\substack{1,\sin x,\sin y,\sin x\sin y\\ p1\text{ invariant}}}$$

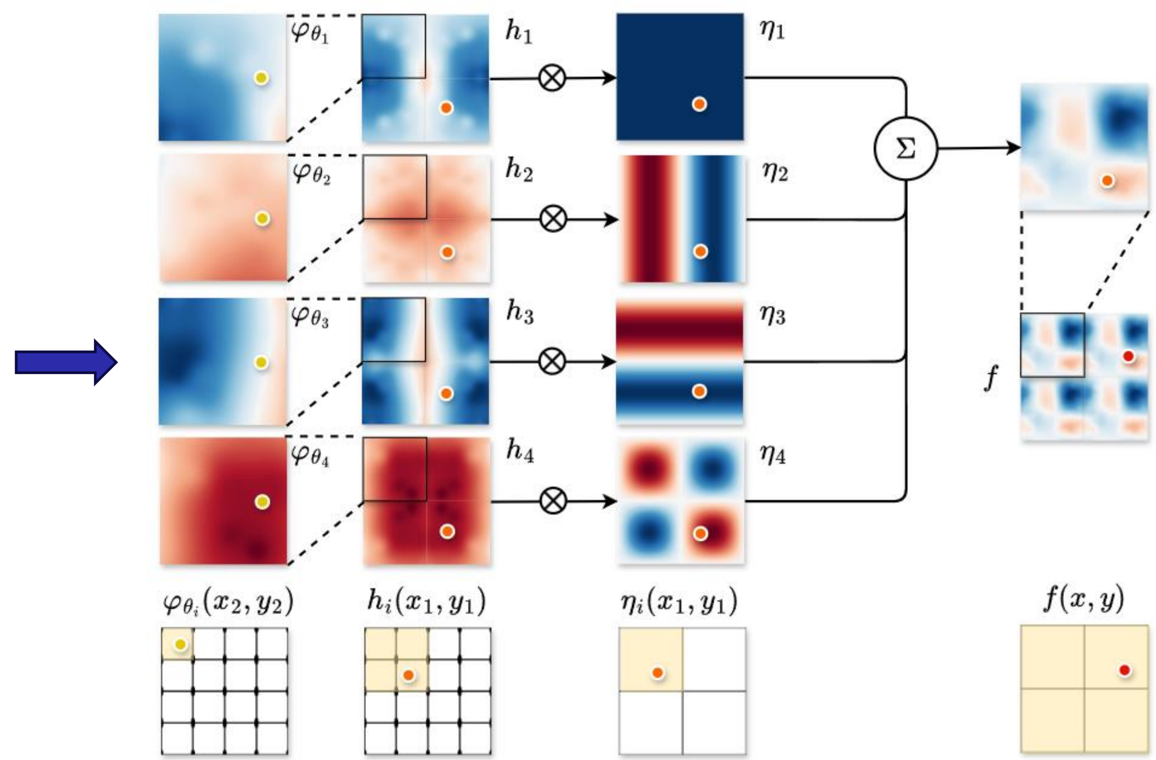
Symmetrization for *p1* Case

Continous symmetrization for affine reflection groups (*p2mm* cases):



Continous symmetrization for *p1* case:

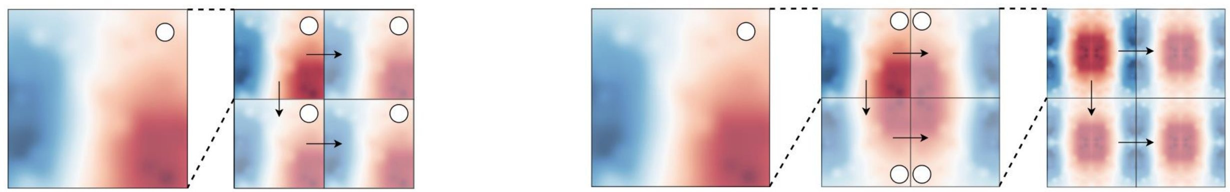
$$f(x,y) = \sum_{i=1}^{4} \underbrace{h_i(x,y)}_{\substack{\text{high-symmetry coeficients} \\ p2mm \text{ invariant}}} \underbrace{\eta_i(x,y)}_{\substack{\text{low-symmetry bases} \\ p1 \text{ invariant}}}$$



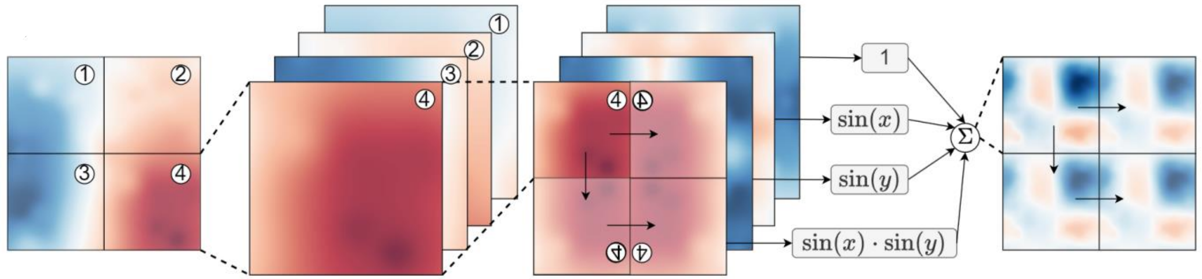
Key Questions

Motivation:

(1) Extension break continuity **(2) Reflection preserve continuity**



(3) Combination of extension with reflection and bases give continuous symmetry



Key questions: To generalize this method to arbitrary planar groups via representation of affine reflection groups, we address the following key questions:

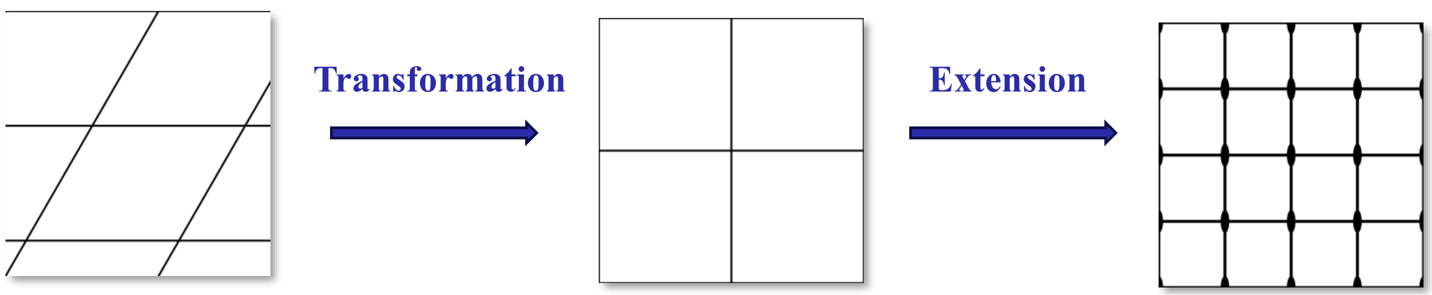
- **Existence of higher symmetry:** For any planar group, does there always exist an affine reflection group serving as a **higher symmetry**?
- **Existence of decomposition:** If exists, does the **decomposition** into high-symmetry coefficients and low-symmetry bases hold for any continuous function?
- **Computation of bases:** How do we explicitly **compute** the low-symmetry bases?

Existence of higher Symmetry

Counterexample (constraint of reflection symmetry): A generic parallelogram lattice cannot support higher affine reflection group symmetry.

Extension up to conjugation:

> **Theorem.** *Any planar group G is conjugate to a subgroup of some affine reflection group W_a, i.e., there exists $A \in GL(2)$ such that $AGA^{-1} \subseteq W_a$.*



p1 (Oblique Lattice) — p1 (Rectangular Lattice) — p2mm (Rectangular Lattice)

Answer:

- Q1: For any planar group, does there always exist an affine reflection group serving as a higher symmetry?
- **A1: Yes, up to conjugation by an invertible linear transformation.**

Existence of Decomposition

Difficult of the problem: Due to the algebraic structure of continuous functions, the existence of exact decomposition of continuous function is hard to answer.

Approximation Theorem:

Theorem. *Let* $r = [W_a : G]$*, and let* Ω *denote the unit cell. Then there exist* r *fixed* G*-invariant bases* $\eta_1, \ldots, \eta_r$ *such that for any* $f \in C_G(\mathbb{R}^2)$ *and any* $\epsilon > 0$*, there exist* $h_1, \ldots, h_r \in C_{W_a}(\mathbb{R}^2)$ *satisfying*

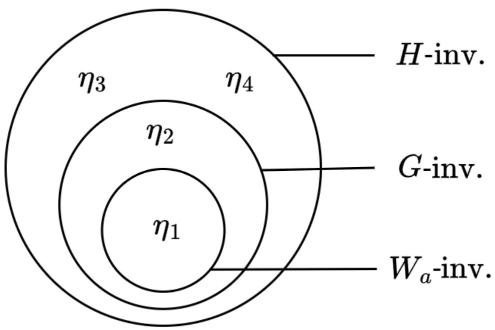
$$\int_\Omega \left| f(\mathbf{x}) - \sum_{i=1}^{r} h_i(\mathbf{x})\, \eta_i(\mathbf{x}) \right| \mathrm{d}\mathbf{x} \; < \; \epsilon.$$

Answer:

- Q2: Does the decomposition into high-symmetry coefficients and a low-symmetry bases always hold for any continuous function?
- **A2: Yes, in the sense of approximation.**

Computation of Bases [7]

Computation using existing results: Result of symmorphic planar groups can be found in Kim et al.



$H \subset G \subset W_a, [G : H] = 2$

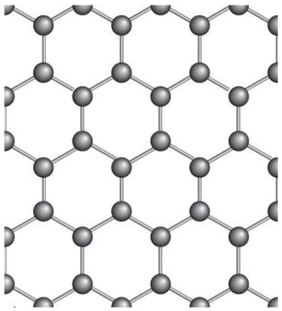
Lattice	G	W_a	**a**	**b**	η_2	η_3	η_4	r
Oblique	$p1$		**a**	**b**	s_1	s_2	$s_1 s_2$	4
Oblique	$p2$		**a**	**b**	$s_1 s_2$	–	–	2
Rectangular	pm		**a**	**b**	s_1	–	–	2
Rectangular	pg		**a**	**b**/2	$s_1 c_2$	$s_1 s_2$	$c_2 s_2$	4
Rectangular	cm	$p2mm$	**a**/2	**b**/2	$c_1 c_2$	$c_1 s_2$	$c_2 s_2$	4
Rectangular	$p2mm$		**a**	**b**	–	–	–	1
Rectangular	$p2mg$		**a**/2	**b**	$s_1 s_2$	–	–	2
Rectangular	$p2gg$		**a**/2	**b**/2	$c_1 c_2$	$c_1 s_1 s_2$	$s_1 c_2 s_2$	4
Rectangular	$c2mm$		**a**/2	**b**/2	$c_1 c_2$	–	–	2
Square	$p4$		**a**	**b**	$(c_1 - c_2) s_1 s_2$	–	–	2
Square	$p4gm$	$p4mm$	**a**/2	**b**/2	$c_1 c_2$	$(c_1 - c_2) s_1 s_2$	$(c_1 - c_2) c_1 c_2 s_1 s_2$	4
Square	$p4mm$		**a**	**b**	–	–	–	1
Hexagonal	$p3$		**a**	**b**	ϕ_1^{-+}	ϕ_2^{--}	$\phi_3^{+-} = \phi_1^{-+} \phi_2^{--}$	4
Hexagonal	$p3m1$		**a**	**b**	ϕ_1^{-+}	–	–	2
Hexagonal	$p31m$	$p6mm$	**a**	**b**	ϕ_2^{--}	–	–	2
Hexagonal	$p6$		**a**	**b**	ϕ_3^{+-}	–	–	2
Hexagonal	$p6mm$		**a**	**b**	–	–	–	1

Answer:

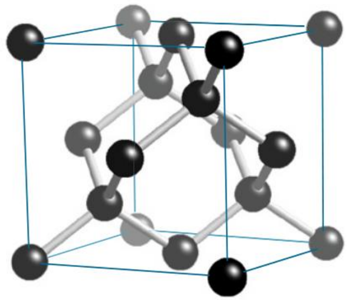
- Q3: How do we explicitly compute the low-symmetry bases?
- **A3: By look up the table.**

Extension to n-Dimensional Space [8]

Crystallographic groups: a crystallographic group is a discrete subgroup of n-dimensional Euclidean group that contains n linearly independent translations.



Graphite (n = 2)
Planar Group *p6mm*



Diamond (n = 3)
Space Group $Fd\bar{3}m$

Answer to key questions:

- Q1: For any crystallographic group, does there always exist an affine reflection group serving as a higher symmetry?
- **A1: Yes (up to conjugation), if $n \leq 3$; no, if $n > 3$.**

- Q2: Does the decomposition into high-symmetry coefficients and a low-symmetry bases always hold for any continuous function?
- **A2: No; but yes (in the sense of approximation), if the crystallographic group is conjugate to a subgroup of an affine reflection group (e.g. $n \leq 3$).**

- Q3 (open): How do we explicitly compute the low-symmetry bases?

Generation with Physical Constraint

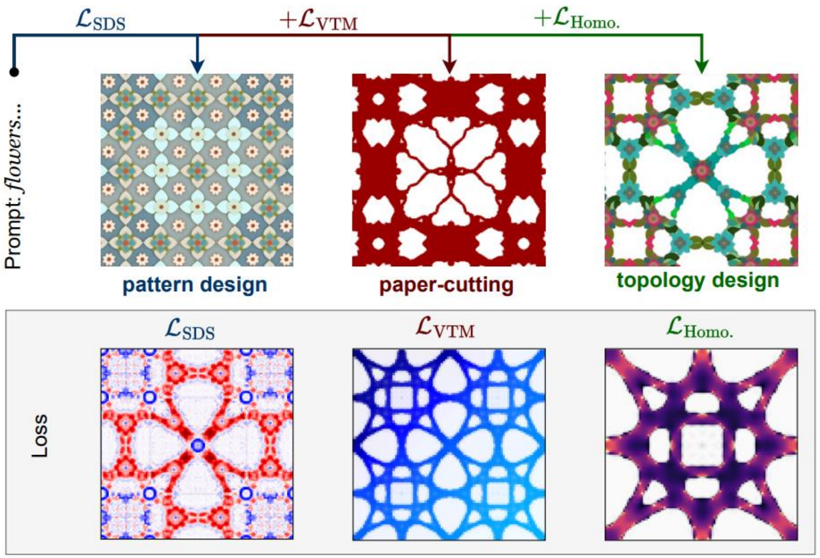
Pattern design (visual semantic constraints): symmetric images that align with given text.

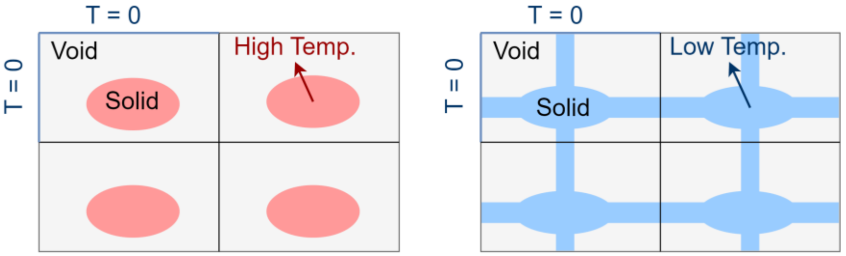
$$\nabla_\theta \mathcal{L}_{\mathrm{SDS}} \;\propto\; \mathbb{E}_{t,\epsilon}\left[\left(\widehat{\epsilon}_\phi(z_t, t, y) - \epsilon\right)\, \nabla_\theta z_\theta\right]$$

Paper-cutting design (visual + connectivity constraints): connected, **symmetric** binary masks subject to **volume** and **text** constraints.

$$\mathcal{L}_{\mathrm{VTM}} = \left(\frac{1}{|\Omega|}\int_\Omega t(\rho_\theta)(\mathbf{x})^p \mathrm{d}\mathbf{x}\right)^{1/p}$$

$$\mathcal{L}_{\mathrm{vol}} = \left(\frac{1}{|\Omega|}\int_\Omega \rho_\theta \,\mathrm{d}\mathbf{x} - \rho_0\right)^2$$





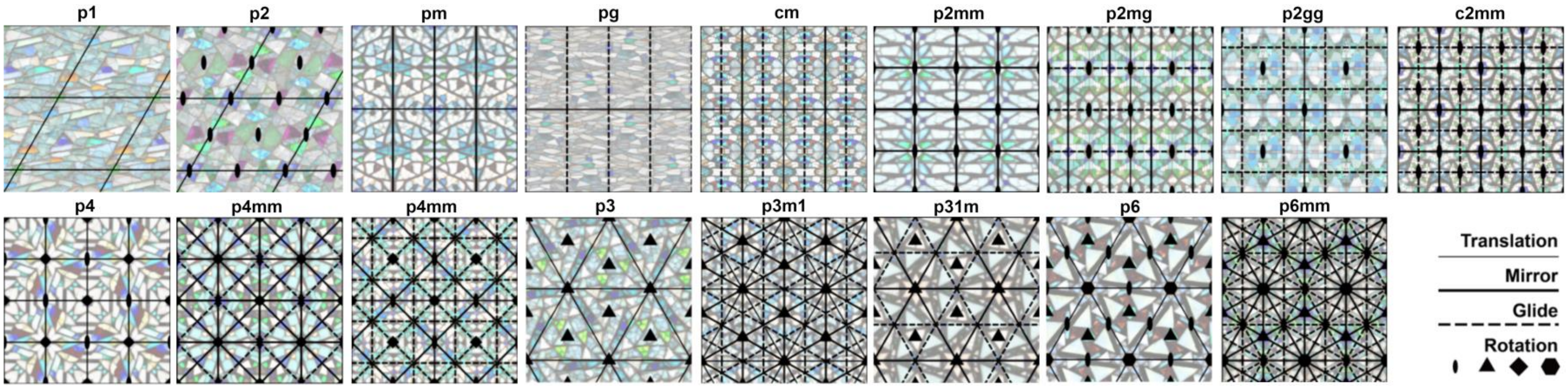
Topology Design (visual + connectivity + mechanical constraints): Generating **connected**, **symmetric** binary masks subject to **volume** and **mechanical** constraints, alongside **text**-aligned images.

$$\mathcal{L}_{\mathrm{Homo}} = -c(\rho_\theta)$$

$$c = E^H_{11} + E^H_{12} + E^H_{21} + E^H_{22}$$

Experiment: Pattern Design

Visualization: 1 prompt (*stained-glass mosaic fragments...*), 17 planar groups

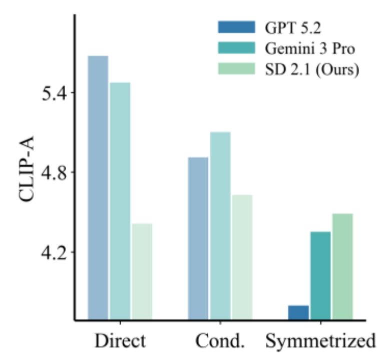


Comparison with other symmetrization:
17 prompts, same planar group

Method	$n = 64$	$n = 128$	$n = 256$
BP ($\alpha = 0.25$)	3.81	3.07	3.46
BP ($\alpha = 0.50$)	3.98	3.38	3.61
BP ($\alpha = 1.00$)	4.05	3.42	3.48
Ours	**4.30**	**4.20**	**3.99**

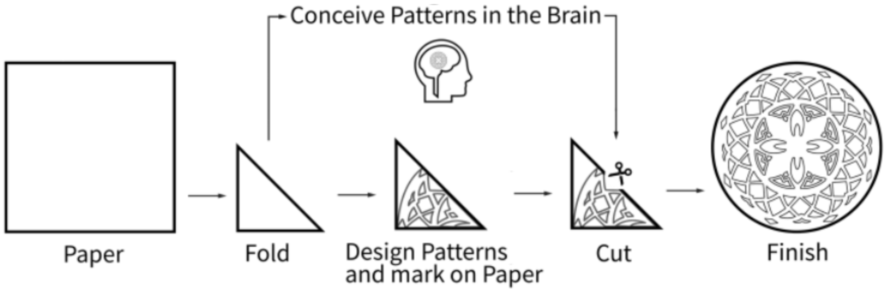
Comparison with text-conditioned generation:
17 prompts, 17 planar groups

- **Direct Generation:** Models generate images directly from the text
- **Conditional Generation:** Generate images with symmetry control
- **Post-Symmetrization:** Fit images with symmetric parameterization using MSE loss

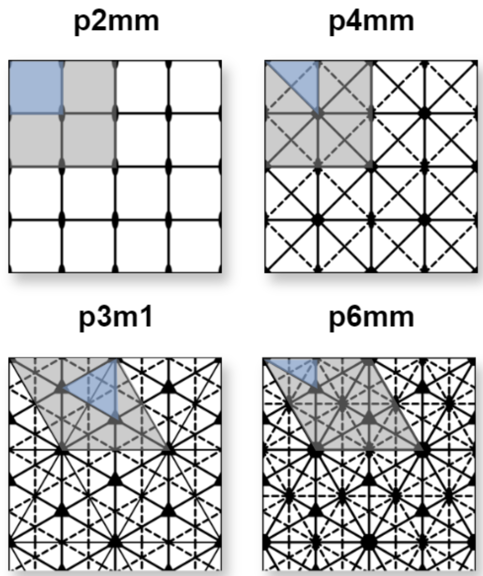


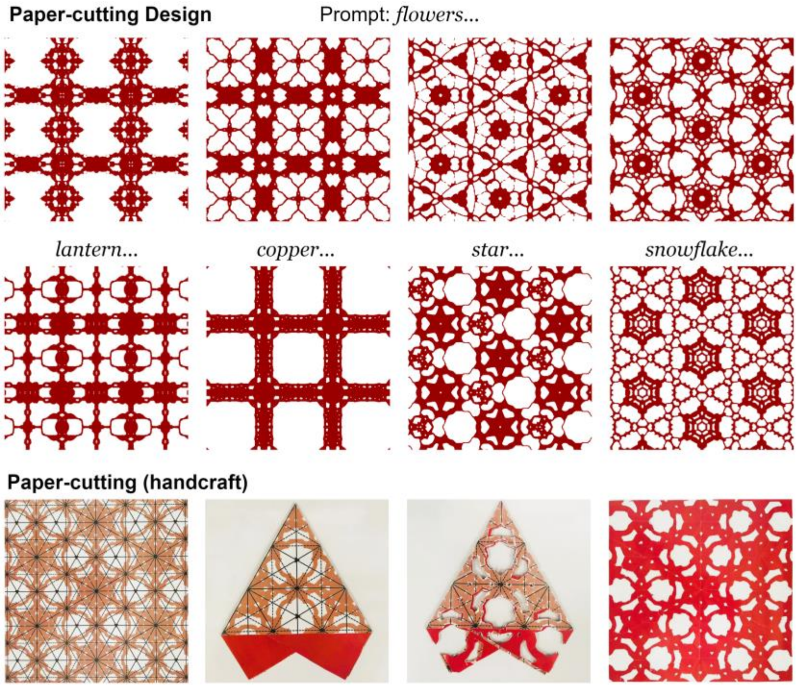
Experiment: Paper-Cutting Design [9]

Paper-cutting design:
- 1 prompt, 4 planar groups
- 4 prompt, 4 planar groups
- Handcraft



Planar Affine Reflection Group

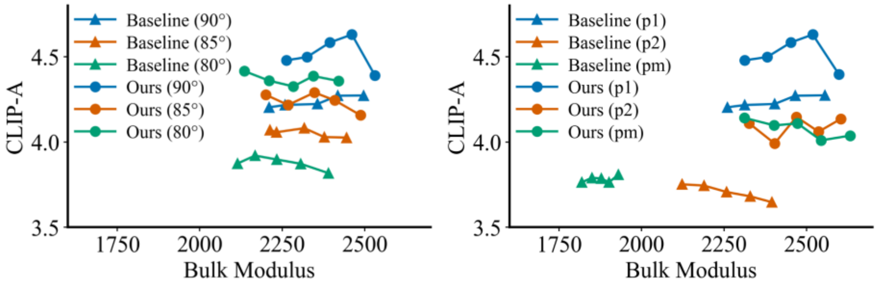


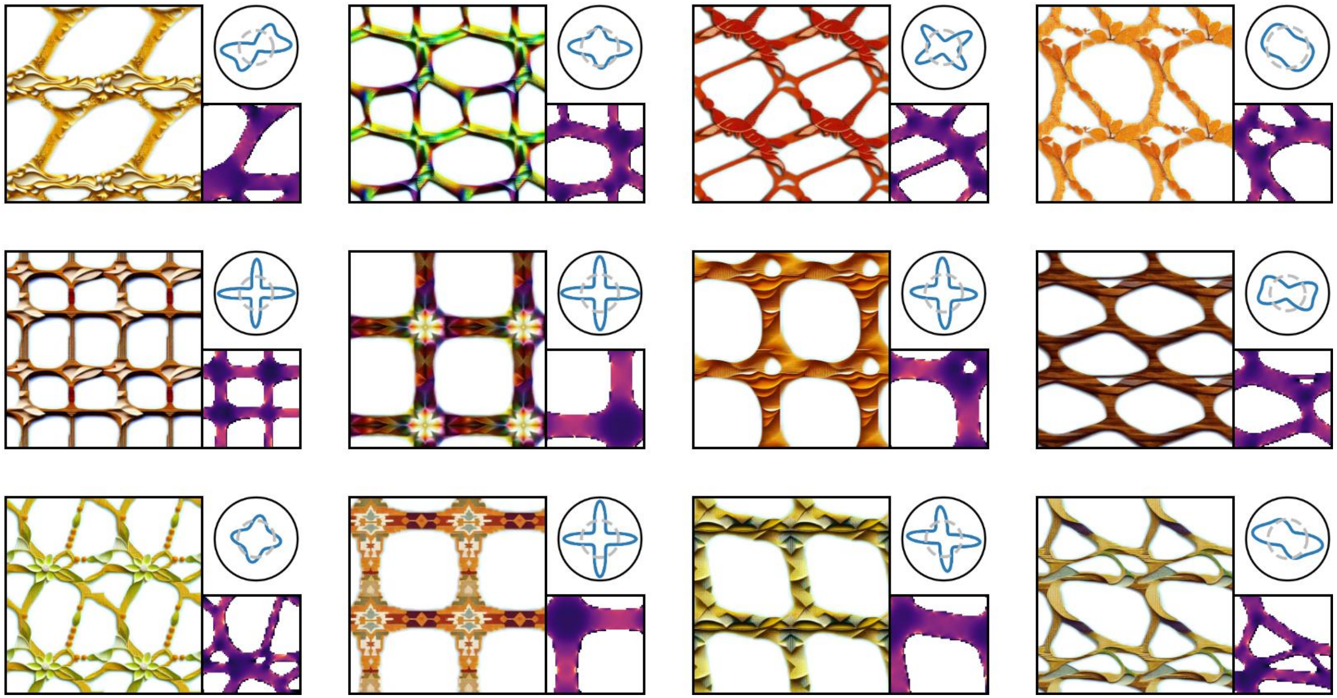


Experiment: Topology Design

Topology design (comparison with other loss design): 12 prompts

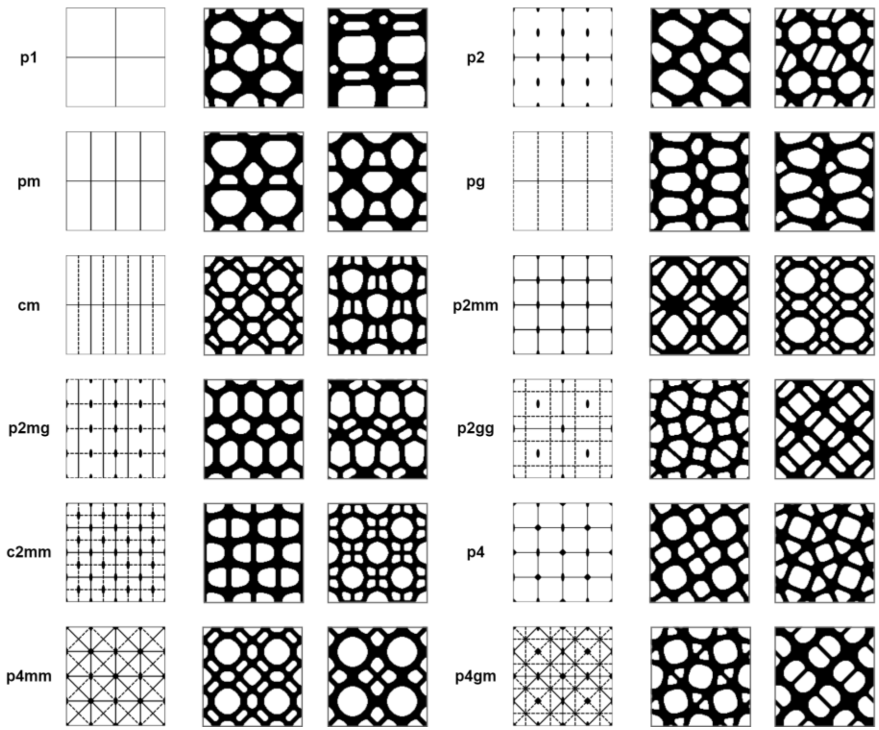
- Differ planar groups
- Same planar groups, differ lattice

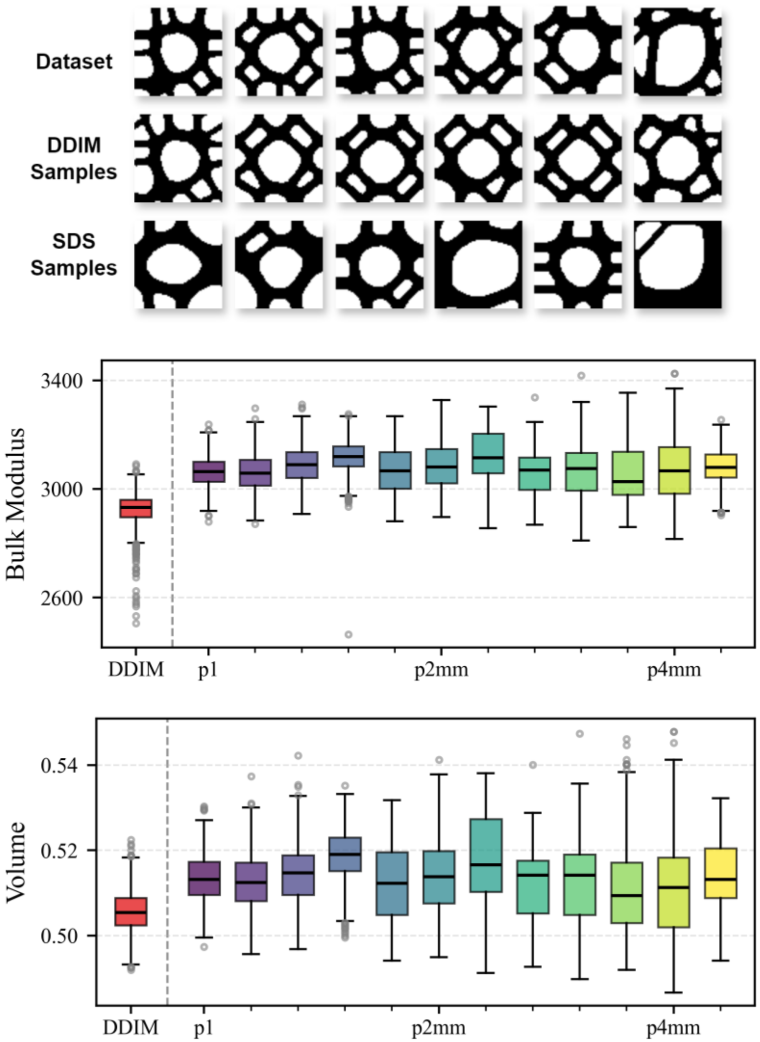




Experiment: Metamaterial Design

Zero-shot generation: Generate symmetric binary mask with pretrained diffusion model and SDS loss (similar to pattern design).





Reference

[1] M. C. Escher's Symmetry Work 16 © 2007 The M.C. Escher Company-Holland. All rights reserved. www.mcescher.com

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