Probabilistic Generating Circuits - Demystified

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Tradeoff between expressiveness and tractability. [Rot96].

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Motivation for our work

- Where does this power of PGCs comes from?
- How powerful they truly are?

Source of power of PGCs*

The power of PGCs comes from negative weights, as PCs with negative weights (*nonmonotone PCs*) subsume PGCs.

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Power of nonmonotone PCs

Nonmonotone PCs computing set-multilinear polynomials support tractable marginalization over categorical variables of an arbitrary image size.

*Proven independently in [BZV24].

DPP example

$$L = \begin{array}{cccc} X_1 & X_2 & X_3 & \\ 4 & 1 & 3 & X_1 \\ 1 & 5 & 2 & X_2 \\ 3 & 2 & 8 & X_3 \end{array}$$

$$Pr(X_1 = 0, X_2 = 1, X_3 = 1)$$

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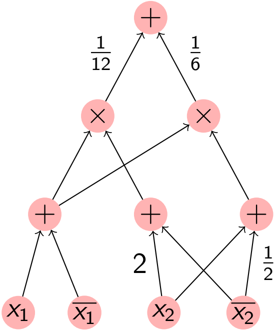
Allow tractable marginalization by storing probabilities as determinants of submatrices.

PC and PGC example

Pr	$X_1 = 0$	$X_1 = 1$
$X_2 = 0$	$\frac{1}{6}$	$\frac{1}{6}$
$X_2 = 1$	$\frac{1}{3}$	$\frac{1}{3}$



$$f = \frac{1}{6}\overline{x_1}\,\overline{x_2} + \frac{1}{3}\overline{x_1}x_2 + \frac{1}{6}x_1\overline{x_2} + \frac{1}{3}x_1x_2$$

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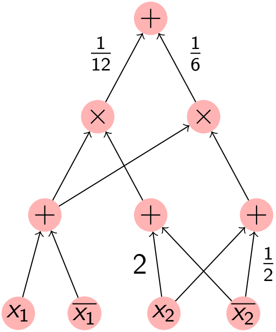

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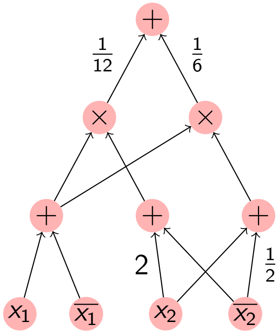

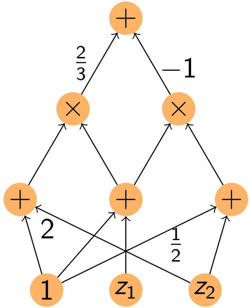

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$$f = \frac{1}{6} + \frac{1}{3}z_2 + \frac{1}{6}z_1 + \frac{1}{3}z_1 z_2$$

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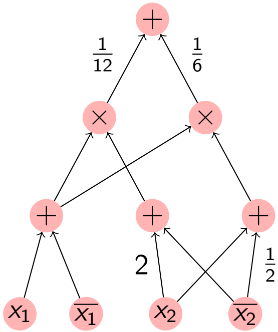

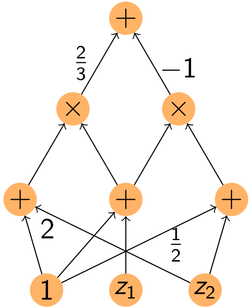

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Tractable marginalization for binary variables.

Separation between PCs and PGCs

Theorem

PGCs over binary variables can be simulated by nonmonotone PCs with only polynomial overhead in size.

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Proof Idea.

- Let a PGC compute $f(z_1, ..., z_n)$. Define the polynomial g as

$$g(x_1, \overline{x_1}, ..., x_n, \overline{x_n}) = f(\frac{x_1}{\overline{x_1}}, \frac{x_2}{\overline{x_2}}, ..., \frac{x_n}{\overline{x_n}}) \cdot \prod_{i=1}^{n} \overline{x_i}.$$

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- g would be *multilinear* and *homogenous* with $deg(g) = n$
- Multiplicative term only increases size of g by $O(n)$.
- To remove division gates, use [Str73]'s result to eliminate them in polynomial overhead.

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Remark: Similar result follows for ternary variables, if we allow marginalization over subsets of images.

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- **Definition (Set-multilinear)**: Let $V = Y_1 \sqcup Y_2 \sqcup ... \sqcup Y_k$. A polynomial p on a set of variables V is set multilinear, **if** every monomial of p contains exactly one variable from each Y_i for all i.

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- Each X_i represented by partition $z_{i,0}, \ldots, z_{i,d-1}$ which are inputs to C.

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  What happens to the power of DPPs under simple pre- and post-processing?
- We show that this question is **Hard** to answer.

Theorem

Any arithmetic formula can be represented as an affine projection of a *DPP*.

Interpretation: If there is a PGC that cannot be written as affine projection of a DPP, then we separate algebraic formulas and circuits.

Conclusions and Future Work

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Future Directions

- Are there efficient marginalisation algorithms for categorical distributions which are not set-multilinear?
- What is the most general class of models that represents tractable distributions?

References

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Thank You!

Questions?