Evaluation of Trajectory Distribution Predictions with Energy Score

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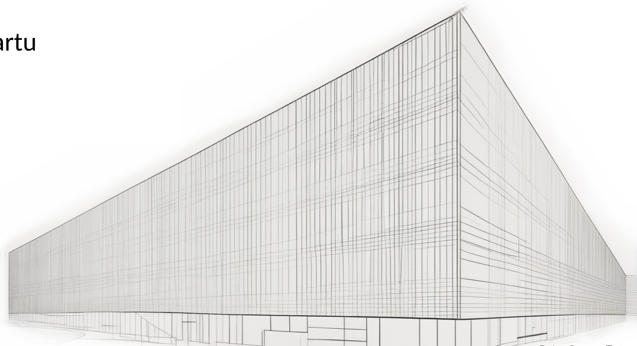
Investing in your future



Delta Center, Tartu

Table of Contents

- Introduction
- Methodology
- Experiments and Results
- Conclusion

Motivation

- **Inherent uncertainty** in the movement of agents

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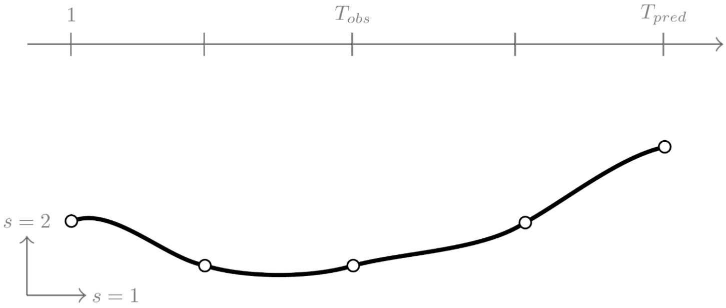
- Estimating uncertainty is vital for **safe and reliable planning**

Trajectory



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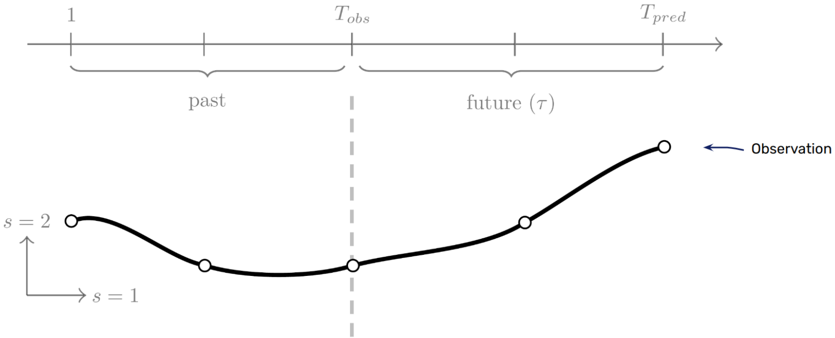
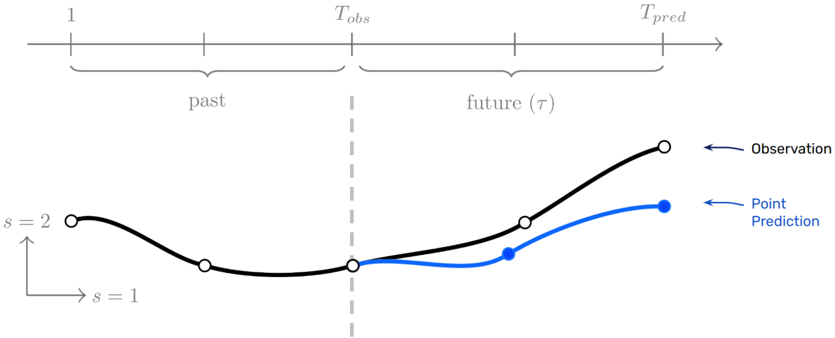


$$\mathbf{y}_i^{\text{past}} = (y_i^{-T_{obs}+1}, \ldots, y_i^0), \quad \mathbf{y}_i^{\tau} = (y_i^1, \ldots, y_i^T) \quad \text{where} \quad y_i^t \in \mathbb{R}^S \quad \text{and} \quad S = \{1, \mathbf{2}, \ldots\}.$$

Trajectory Prediction



$$\mathbf{y}_i = (y_i^{t=1}, \ldots, y_i^{t=T}) \sim \mathbf{Y}_i \quad \text{where} \quad \mathbf{y}_i \in \mathbb{R}^{T \times S} \quad \text{and} \quad y_i^t \in \mathbb{R}^S$$

$$\mathbf{x}_i = (x_i^{t=1}, \ldots, x_i^{t=T}) \sim \mathbf{X}_i \quad \text{where} \quad \mathbf{x}_i \in \mathbb{R}^{T \times S} \quad \text{and} \quad x_i^t \in \mathbb{R}^S$$

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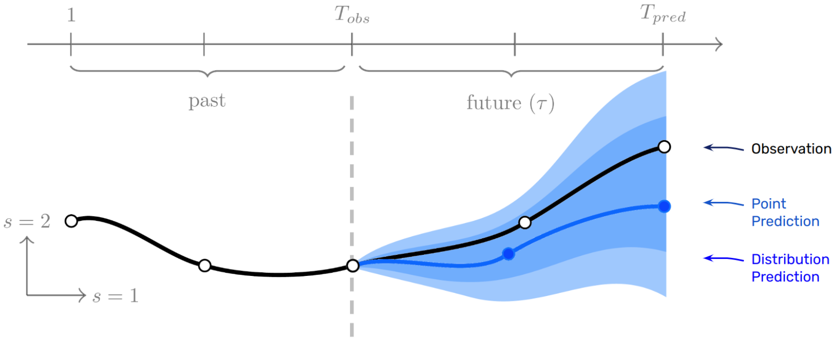

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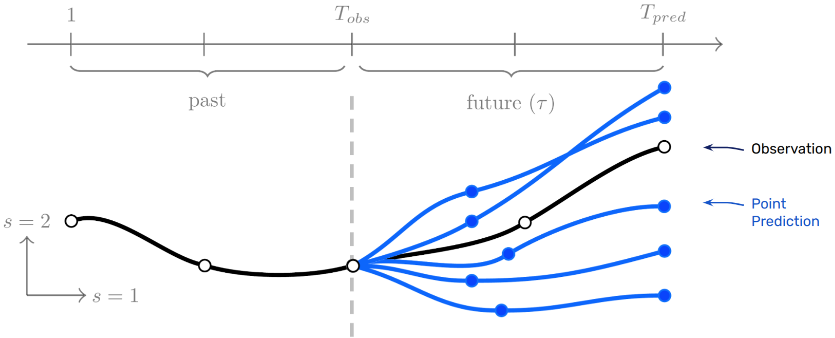

$$\mathbf{x}_{i,1}, \ldots, \mathbf{x}_{i,K} \sim \mathbf{X}_i \quad \text{where} \quad \mathbf{x}_{i,k} \in \mathbb{R}^{T \times S}, \quad x^t_{i,k} \in \mathbb{R}^S$$

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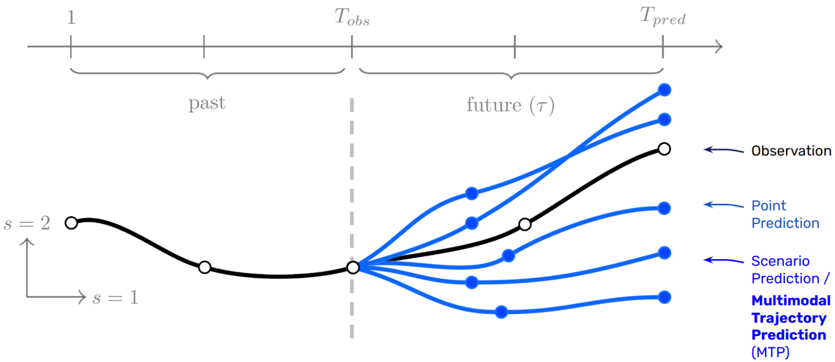

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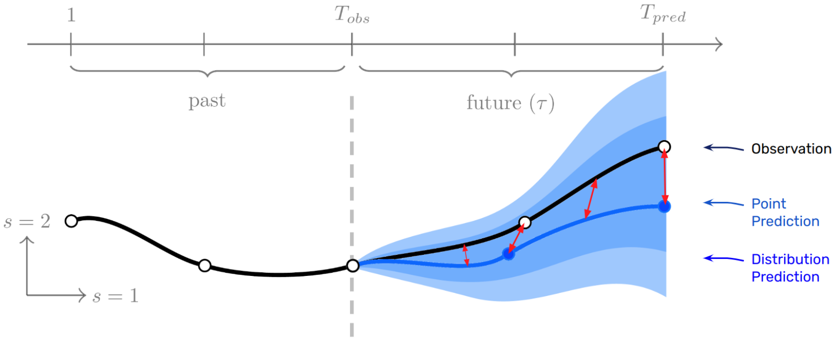
Trajectory Distribution Evaluation



Trajectory Distribution Evaluation

It measures the distance between the Predicted and ground truth distributions.

$$distance(\mathbf{F}_{\mathbf{X}_i}, \mathbf{F}_{\mathbf{Y}_i})$$

Where
$\mathbf{F}_{\mathbf{X}_i}$ is the CDF of the predicted.
$\mathbf{F}_{\mathbf{Y}_i}$ is the CDF of the ground truth.
i is the index of N instances in the dataset.

Variety Loss

Definition

$$L_{variety}(\mathbf{X}_i, \mathbf{y}_i) = \mathbb{E} \min_{k<K} \|\mathbf{x}_{i,k} - \mathbf{y}_i\|_2$$

Illustration adapted from Thiede, Luca Anthony, and Pratik Prabhanjan Brahma. "Analyzing the variety loss in the context of probabilistic trajectory prediction." Proceedings of the IEEE/CVF International Conference on Computer Vision. 2019.

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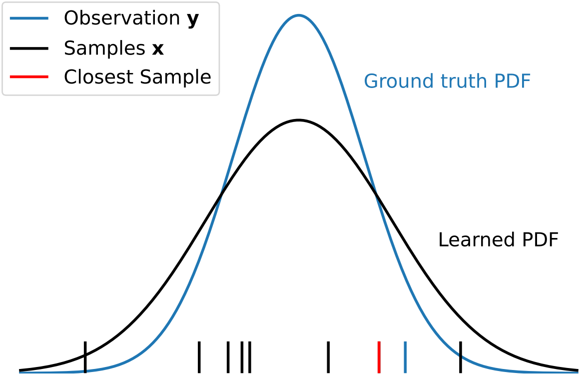

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- Variety loss employed as a complementary loss
- Variety loss found its way as an evaluation metric (Minimum of N)
- minFDE/minADE are common instances of Minimum of N

Common Metrics for Single Trajectory Prediction

Average Displacement Error

$$ADE(\mathbf{x}_i, \mathbf{y}_i) = \frac{1}{T} \sum_{t=1}^{T} \|x_i^t - y_i^t\|_2$$

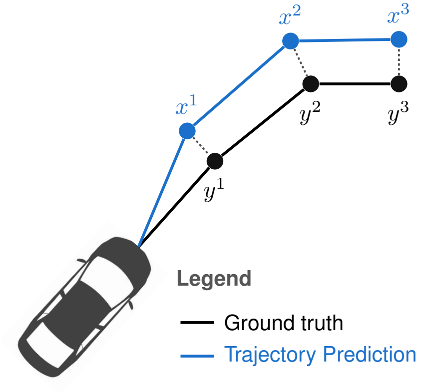
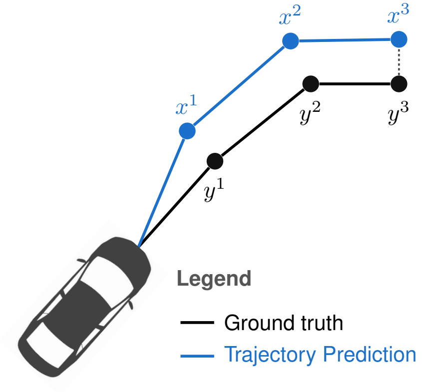


Illustration inspired by Boris, Ivanovic, and M. Pavone. "Rethinking trajectory forecasting evaluation." arXiv preprint arXiv:2107.10297 (2021).

Common Metrics for Single Trajectory Prediction

Final Displacement Error

$$FDE(\mathbf{x}_i, \mathbf{y}_i) = \|x_i^T - y_i^T\|_2$$



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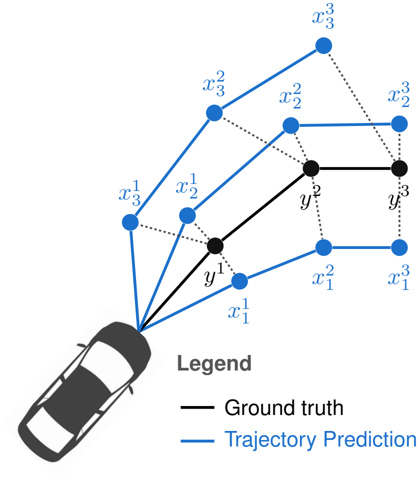
ADE/FDE on Multimodal Trajectory Prediction (MTP)

Average Displacement Error on MTP

$$ADE(\mathbf{X}_i, \mathbf{Y}_i) = \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\|X_i^t - Y_i^t\|_2\right]$$

$$\widehat{ADE}(\mathbf{x}_i, \mathbf{y}_i) = \frac{1}{KT}\sum_{k=1}^{K}\sum_{t=1}^{T}\|x_{i,k}^t - y_i^t\|_2$$



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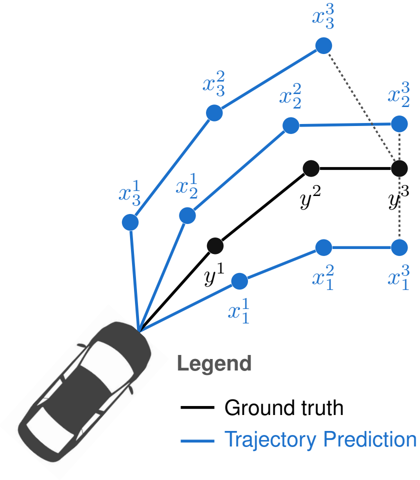
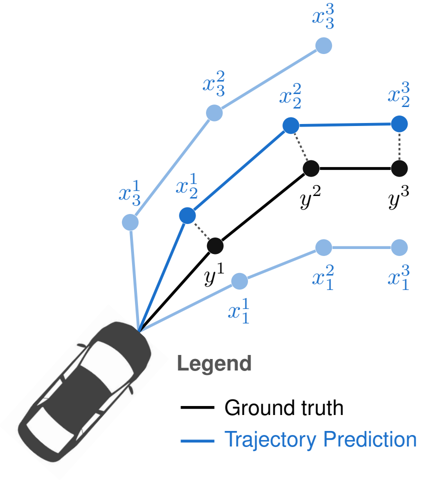


Common instances of Minimum of N (MoN)

Minimum Average Displacement Error

$$minADE(\mathbf{X}_i, \mathbf{Y}_i) = \mathbb{E}\left[\min_k \frac{1}{T}\sum_{t=1}^{T} \|X_{i,k}^t - Y_i^t\|_2\right]$$

$$\widehat{minADE}(\mathbf{x}_i, \mathbf{y}_i) = \min_k \frac{1}{T}\sum_{t=1}^{T} \|x_{i,k}^t - y_i^t\|_2$$

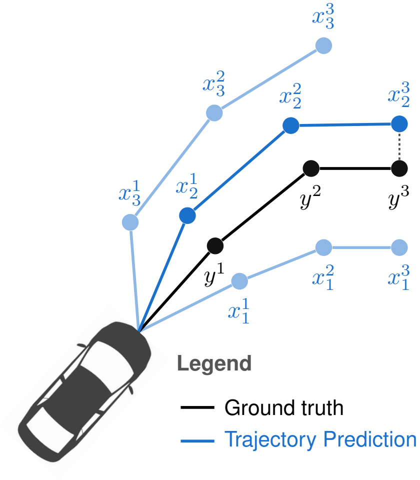


Common instances of Minimum of N (MoN)

Minimum Final Displacement Error

$$minFDE(\mathbf{X}_i, \mathbf{Y}_i) = \mathbb{E}\left[\min_k \|X_{i,k}^T - Y_i^T\|_2\right]$$

$$\widehat{minFDE}(\mathbf{x}_i, \mathbf{y}_i) = \min_k \|x_{i,k}^T - y_i^T\|_2$$



"L-lowest of N" (LoN)

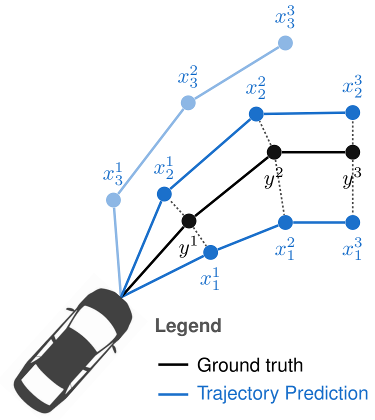
LoN as a more general form of "Minimum of N"

$$ADE_{(L)}(\mathbf{X}_i, \mathbf{Y}_i) = \mathbb{E} \min_{\substack{\{k_1,\dots,k_L\} \\ k_i \neq k_j}} \frac{1}{LT} \sum_{l=1}^{L} \sum_{t=1}^{T} DE(X^t_{i,k_l}, Y^t_i)$$

$$FDE_{(L)}(\mathbf{X}_i, \mathbf{Y}_i) = \mathbb{E} \min_{\substack{\{k_1,\dots,k_L\} \\ k_i \neq k_j}} \frac{1}{L} \sum_{l=1}^{L} DE(X^T_{i,k_l}, Y^T_i)$$

$$ADE_{(L=1)} \equiv minADE \quad , \quad FDE_{(L=1)} \equiv minFDE$$

$$ADE_{(L=K)} \equiv ADE \quad , \quad FDE_{(L=K)} \equiv FDE$$



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$$f_{m_i}(x) = \frac{(f_Y)^{m_i}}{C_{m_i}} \quad \text{where} \quad m_2(K_2) < m_1(K_1) \quad \text{when} \quad K_2 > K_1$$

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Implications of using Variety Loss or MoN

- **Loss function:** less sharp (high-variance density) or too sharp (low-variance density) depending on the value of K and dimensionality of the target distribution

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- **Evaluation:** probabilistic calibration not respected
- **Application:** Cost induced on the prevalent or extreme events

Table of Contents

2 Methodology

▶ Introduction

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Our contribution

1. Study MoN as a scoring rule

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2. Propose energy score-based evaluation

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1. Study MoN as a scoring rule
2. Propose energy score-based evaluation
3. Different ways energy score can be employed for the evaluation

Proper Scoring Rule

Definition

A (negatively-oriented) strictly proper scoring rule $\mathbf{S}$ maps a probability distribution $\mathbf{F_X}$ and an observation $\mathbf{y}$ to a real number, i.e., $\mathbf{S}(\mathbf{F_X}, \mathbf{y}) \in \mathbb{R}$. The expected value of $\mathbf{S}(\mathbf{F_X}, .)$ under $\mathbf{F_Y}$, is written as $\mathbf{S}(\mathbf{F_X}, \mathbf{F_Y}) = \mathbb{E}_{\mathbf{y} \sim \mathbf{F_Y}}[\mathbf{S}(\mathbf{F_X}, \mathbf{y})]$.

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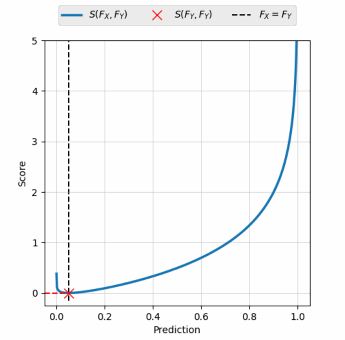

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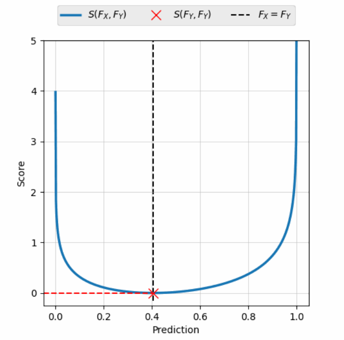

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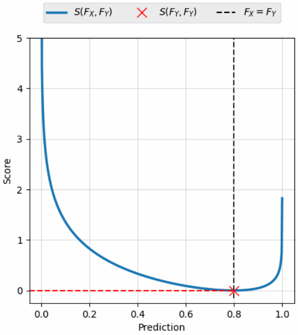

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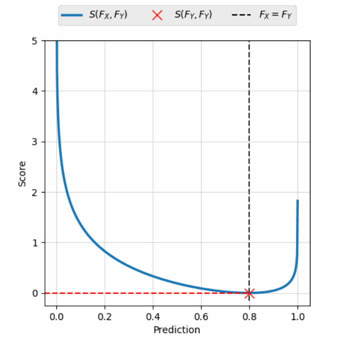

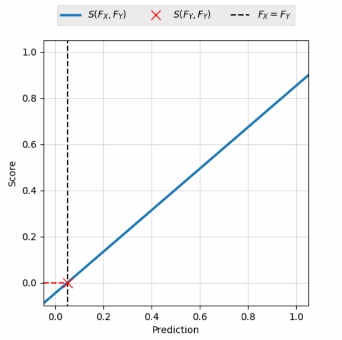

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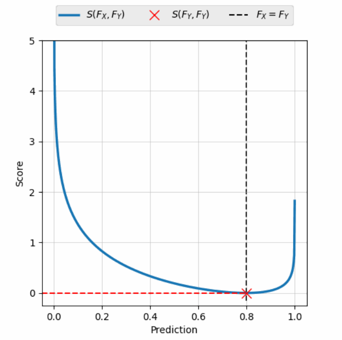

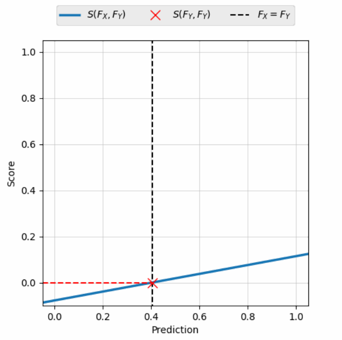

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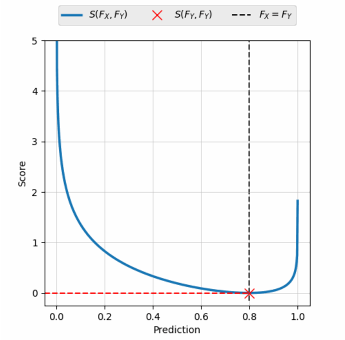

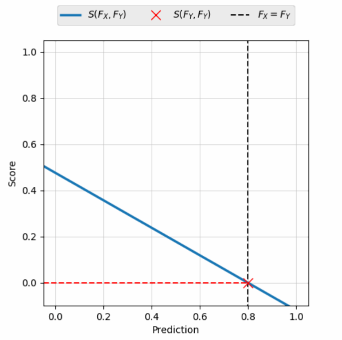

MoN as a Scoring Rule

Proposition 4.1

Average Displacement Error $ADE(\mathbf{X}_i, \mathbf{Y}_i)$ *is improper, meaning there exist distributions* $\mathbf{F}_{\mathbf{X}_i}$ *and* $\mathbf{F}_{\mathbf{Y}_i}$*, for which* $ADE(\mathbf{X}_i, \mathbf{Y}_i) < ADE(\mathbf{Y}_i, \mathbf{Y}_i)$.

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Choosing $X^t \sim N(\mu_{X^t}, \sigma^2_{X^t})$ such that $\mu_{X^t} = \mu_{Y^t}$ and $\sigma^2_{X^t} \to 0$, we get that

$$ADE(\mathbf{X}, \mathbf{Y}) \to \frac{1}{T} \sum_{t=1}^{T} \sqrt{\sigma^2_{Y^t}} \frac{\sqrt{2}}{\sqrt{\pi}} < \frac{1}{T} \sum_{t=1}^{T} \sqrt{\sigma^2_{Y^t} + \sigma^2_{Y^t}} \frac{\sqrt{2}}{\sqrt{\pi}} = ADE(\mathbf{Y}, \mathbf{Y}). \quad \square$$

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Because ADE is improper, FDE is improper too.

Proposition 4.2

L-lowest Average Displacement Error $ADE_{(L)}(\mathbf{X}_i, \mathbf{Y}_i)$ *is improper for any values of* $L \leq 2$, *meaning there exist distributions* $\mathbf{F}_{\mathbf{X}_i}$ *and* $\mathbf{F}_{\mathbf{Y}_i}$, *for which* $ADE_{(L)}(\mathbf{X}_i, \mathbf{Y}_i) < ADE_{(L)}(\mathbf{Y}_i, \mathbf{Y}_i)$.

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For the Proof, refer to the paper. We provide proof for $\mathbf{X} = X_1, \ldots, X_K \overset{i.i.d}{\sim} Ber(p_X)$ and a random variable $Y \sim Ber(p_Y)$ when $S = 1$ and $T = 1$.

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- It is strictly proper only when $K = 2$.

MoN as a Scoring Rule

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- It is strictly proper only when $K = 2$.
- When $K = 1$ and $p_Y > 0.5$, then $p_X = 1$ gives the lowest ADE

MoN as a Scoring Rule

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- When $K = 1$ and $p_Y < 0.5$, then $p_X = 0$ gives the lowest ADE

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- When $K = 1$ and $p_Y < 0.5$, then $p_X = 0$ gives the lowest ADE
- When $K >= 3$ and the $p_Y \neq 0.5$, the lowest ADE is obtained by p_X value that is between p_Y and 0.5.

MoN as a Scoring Rule

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L-lowest Average Displacement Error $ADE_{(L)}(\mathbf{X}_i, \mathbf{Y}_i)$ is improper for any values of $L \leq 2$, meaning there exist distributions $\mathbf{F}_{\mathbf{X}_i}$ and $\mathbf{F}_{\mathbf{Y}_i}$, for which $ADE_{(L)}(\mathbf{X}_i, \mathbf{Y}_i) < ADE_{(L)}(\mathbf{Y}_i, \mathbf{Y}_i)$.

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- It is strictly proper only when $K = 2$.
- When $K = 1$ and $p_Y > 0.5$, then $p_X = 1$ gives the lowest ADE
- When $K = 1$ and $p_Y < 0.5$, then $p_X = 0$ gives the lowest ADE
- When $K >= 3$ and the $p_Y \neq 0.5$, the lowest ADE is obtained by p_X value that is between p_Y and 0.5.
- When $K \to \infty$, the optimal $p_X \to 0.5$.

Proposition 4.4

Let $\mathbf{X}_i \sim \mathbf{F}_{\mathbf{X}_i}$ *of length* K *and* $\mathbf{Y}_i \sim \mathbf{F}_{\mathbf{Y}_i}$. *If* $K \to \infty$, L *is fixed and supp*$(\mathbf{F}_{\mathbf{Y}_i}) \subset$ *supp*$(\mathbf{F}_{\mathbf{X}_i})$ *then* $ADE_{(L)}(\mathbf{X}_i, \mathbf{Y}_i) \to 0$.

For the Proof, refer to the paper.

Energy Score

- Energy Distance [1]

[1] Székely, G.J., Rizzo, M.L.: Energy statistics: A class of statistics based on distances. Journal of Statistical Planning and Inference. 143, 1249–1272 (2013).

Energy Score

- Energy Distance [1]
- Other related measures

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Energy Score

- Energy Distance [1]
- Other related measures
  - Generalization of CRPS [2]

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[2] Gneiting, T., Raftery, A.E.: Strictly Proper Scoring Rules, Prediction, and Estimation. Journal of the American Statistical Association. 102, 359–378 (2007).

Energy Score

- Energy Distance [1]
- Other related measures
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  - Permutational Analysis of Variance (PERMANOVA) [3]

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Energy Score

- Energy Distance [1]
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[1] Székely, G.J., Rizzo, M.L.: Energy statistics: A class of statistics based on distances. Journal of Statistical Planning and Inference. 143, 1249–1272 (2013).

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[3] Anderson, M. (2001). A new method for non-parametric multivariate analysis of variance. Australecology, 26(1), 32–46.

[4] Ramdas A, Trillos NG, Cuturi M. On Wasserstein Two-Sample Testing and Related Families of Nonparametric Tests. Entropy. 2017; 19(2):47.

Definition

$$ES(\mathbf{F}_{\mathbf{X}_i}, \mathbf{y}_i) = \overbrace{\mathbb{E}\|\mathbf{X}_i - \mathbf{y}_i\|_p^\beta}^{ED} - \frac{1}{2}\overbrace{\mathbb{E}\|\mathbf{X}_i - \tilde{\mathbf{X}}_i\|_p^\beta}^{EI} \quad (1)$$

$$\overline{ES} = \frac{1}{N}\sum_{i=1}^{N} ES(\mathbf{F}_{\mathbf{X}_i}, \mathbf{y}_i) \quad (2)$$

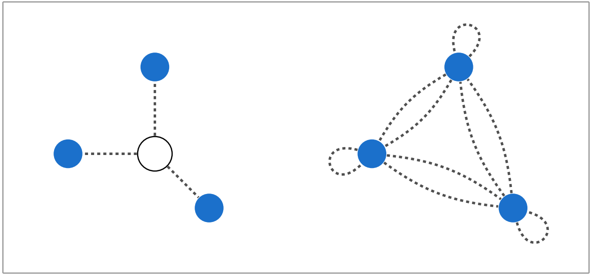
$$\widehat{ES} = \frac{1}{K}\sum_{k=1}^{K}\|x_{i,k} - y_i\|_2 - \frac{1}{2 \cdot K^2}\sum_{k=1}^{K}\sum_{k'=1}^{K}\|x_{i,k} - x_{i,k'}\|_2 \quad \text{where} \quad \beta = 1, \quad p = 2$$

Energy Score Intuition

$$\widehat{ES} = \frac{1}{K}\sum_{k=1}^{K} \|x_{i,k} - y_i\|_2 - \frac{1}{2 \cdot K^2}\sum_{k=1}^{K}\sum_{k'=1}^{K} \|x_{i,k} - x_{i,k'}\|_2$$

Legend

- Observation y
- Prediction x



Energy Score Intuition

$$\widehat{ES} = \frac{1}{3}\sum_{k=1}^{3} \|x_{i,k} - y_i\|_2 - \frac{1}{2 \cdot 3^2}\sum_{k=1}^{3}\sum_{k'=1}^{3} \|x_{i,k} - x_{i,k'}\|_2$$

Legend

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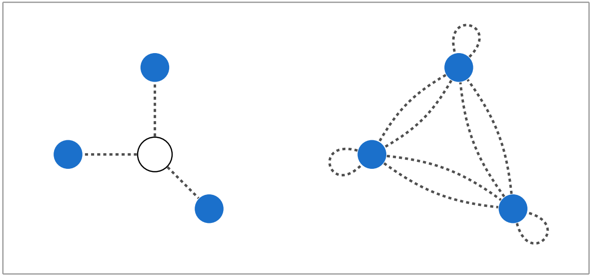
- Prediction x



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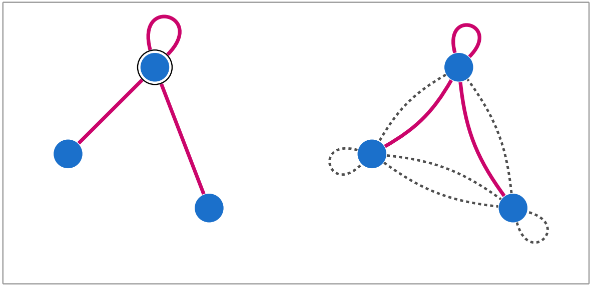
$i = 1$



Legend

- Observation y
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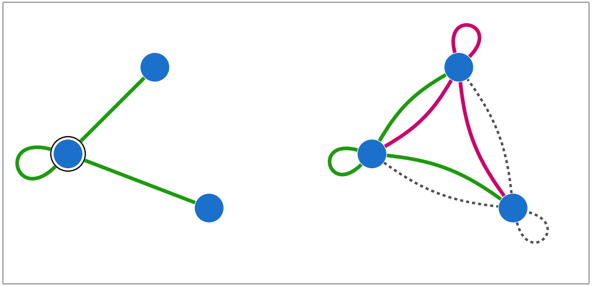
Energy Score Intuition

$$\widehat{ES} = \frac{1}{3}\sum_{k=1}^{3} \|x_{i,k} - y_i\|_2 - \frac{1}{2 \cdot 3^2}\sum_{k=1}^{3}\sum_{k'=1}^{3} \|x_{i,k} - x_{i,k'}\|_2$$

$i = 2$

Legend

- Observation y
- Prediction x



Energy Score Intuition

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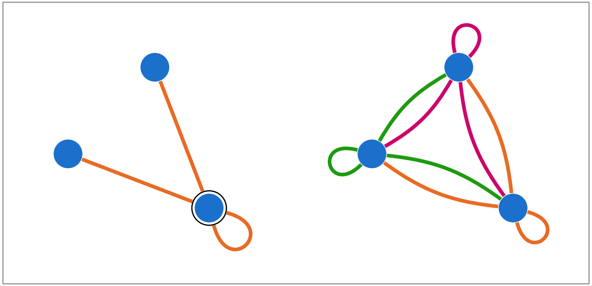
$i = 3$

Legend

Observation y

Prediction x



Energy Score Intuition

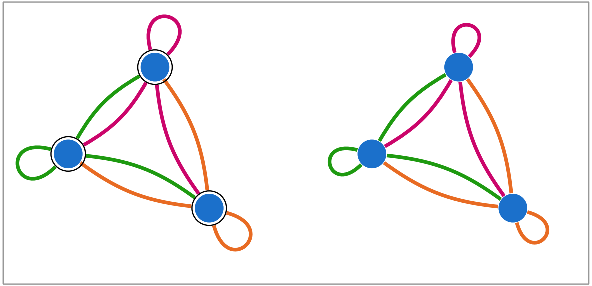
$$\widehat{ES} = \frac{1}{3}\sum_{k=1}^{3} \|x_{i,k} - y_i\|_2 - \frac{1}{2\cdot 3^2}\sum_{k=1}^{3}\sum_{k'=1}^{3} \|x_{i,k} - x_{i,k'}\|_2$$

$$i = \{1, 2, 3\}$$

Legend

Observation y

Prediction x



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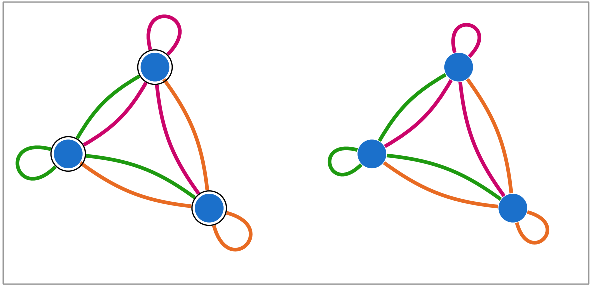
$$\widehat{ES} = \frac{1}{3}\sum_{i=1}^{3}\frac{1}{3}\sum_{k=1}^{3}\|x_{i,k} - y_i\|_2 - \frac{1}{3}\sum_{i=1}^{3}\frac{1}{2\cdot 3^2}\sum_{k=1}^{3}\sum_{k'=1}^{3}\|x_{i,k} - x_{i,k'}\|_2$$

$$i = \{1, 2, 3\}$$

Legend

Observation y

Prediction x



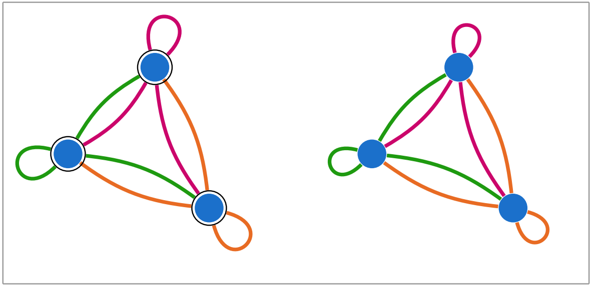
Energy Score Intuition

$$\widehat{ES} = \frac{1}{3^2}\sum_{i=1}^{3}\sum_{k=1}^{3}\|x_{i,k} - y_i\|_2 - \frac{1}{2}\frac{1}{3^2}\sum_{k=1}^{3}\sum_{k'=1}^{3}\|x_{i,k} - x_{i,k'}\|_2$$

$$i = \{1, 2, 3\}$$

Legend

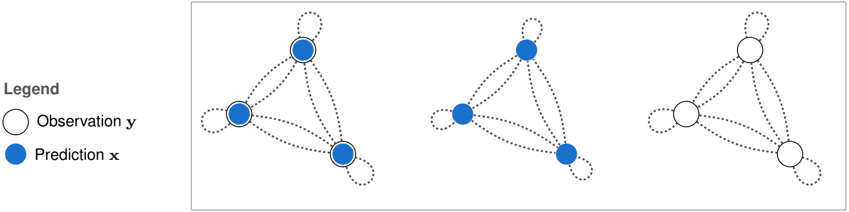
- Observation y
- Prediction x



Energy Score Intuition

$$ED(F_X, F_Y) = \mathbb{E}(\|X - Y\|_p^{\beta}) - \frac{1}{2}\mathbb{E}(\|X - \tilde{X}\|_p^{\beta}) - \frac{1}{2}\mathbb{E}(\|Y - \tilde{Y}\|_p^{\beta})$$

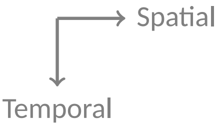
$$\widehat{ED} = \frac{1}{3^2}\sum_{i=1}^{3}\sum_{k=1}^{3}\|x_{i,k} - y_{i,k}\|_2 - \frac{1}{2}\frac{1}{3^2}\sum_{k=1}^{3}\sum_{k'=1}^{3} 3^2 \|x_{i,k} - x_{i,k'}\|_2 - \frac{1}{2}\frac{1}{3^2}\sum_{k=1}^{3}\sum_{k'=1}^{3} 3^2 \|y_{i,k} - y_{i,k'}\|_2$$



Székely, G.J., Rizzo, M.L.: Energy statistics: A class of statistics based on distances. Journal of Statistical Planning and Inference. 143, 1249–1272 (2013).

Energy Score for MTP Evaluation

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$



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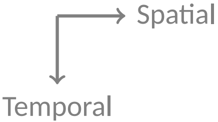
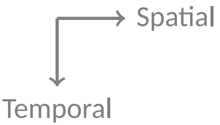


$$dim(\{\mathbf{x}_{i,k}\}_{k=1}^{K}) = K \times T \times S \qquad dim(\mathbf{y}_i) = 1 \times T \times S$$

Energy Score for MTP Evaluation

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$

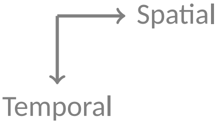


$$dim(\{\mathbf{x}_{i,k}\}_{k=1}^{K}) = K \times T \times S \qquad dim(\mathbf{y}_i) = 1 \times T \times S$$

- Entry-wise (jointly on spatial and temporal)

Energy Score for MTP Evaluation

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$

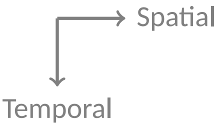


$$dim(\{\mathbf{x}_{i,k}\}_{k=1}^{K}) = K \times T \times S \qquad dim(\mathbf{y}_i) = 1 \times T \times S$$

- Entry-wise (jointly on spatial and temporal)
- Column-wise (marginalized on Spatial)

Energy Score for MTP Evaluation

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$



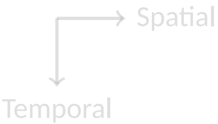
$$dim(\{\mathbf{x}_{i,k}\}_{k=1}^{K}) = K \times T \times S \qquad dim(\mathbf{y}_i) = 1 \times T \times S$$

- Entry-wise (jointly on spatial and temporal)
- Column-wise (marginalized on Spatial)
- Row-wise (marginalized on Temporal)

  remark: common metrics such as minADE are typically temporally marginalized

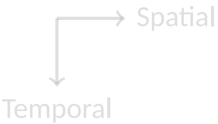
Entry-wise

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$



Entry-wise

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$

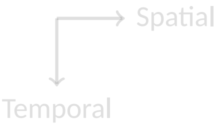


$$ES(\mathbf{F}_{\mathbf{X}_i}, \mathbf{y}_i) = \mathbb{E}_k\left(\|\mathbf{X} - \mathbf{y}\|_p^\beta\right) - \frac{1}{2}\mathbb{E}_k\left(\|\mathbf{X} - \tilde{\mathbf{X}}\|_p^\beta\right)$$

akin to Frobenius distance for $p = 2$.

Entry-wise

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$



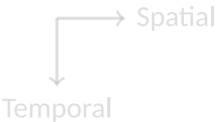
$$ES(\mathbf{F}_{\mathbf{x}_i}, \mathbf{y}_i) = \mathbb{E}_k\left(\|\mathbf{X} - \mathbf{y}\|_p^\beta\right) - \frac{1}{2}\mathbb{E}_k\left(\|\mathbf{X} - \tilde{\mathbf{X}}\|_p^\beta\right)$$

$$\widehat{ES}(\mathbf{x}_i, \mathbf{y}_i) = \frac{1}{K}\sum_{k=1}^{K}\left(\sum_{t=1}^{T}\sum_{s=1}^{S}|x_k^{t,s} - y^{t,s}|^p\right)^{\beta/p} - \frac{1}{2}\frac{1}{K^2}\sum_{k=1}^{K}\sum_{l=1}^{K}\left(\sum_{t=1}^{T}\sum_{s=1}^{S}|x_k^{t,s} - \tilde{x}_l^{t,s}|^p\right)^{\beta/p}$$

akin to Frobenius distance for $p = 2$.

Entry-wise

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \cdots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \cdots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \cdots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \cdots & y^{TS} \end{bmatrix}$$



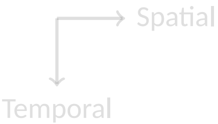
$$ES(\mathbf{F}_{\mathbf{X}_i}, \mathbf{y}_i) = \mathbb{E}_k\left(\|\mathbf{X} - \mathbf{y}\|_p^\beta\right) - \frac{1}{2}\mathbb{E}_k\left(\|\mathbf{X} - \tilde{\mathbf{X}}\|_p^\beta\right)$$

$$\widehat{ES}(\mathbf{x}_i, \mathbf{y}_i) = \frac{1}{K}\sum_{k=1}^{K}\left(\sum_{t=1}^{T}\sum_{s=1}^{S}|x_k^{t,s} - y^{t,s}|^p\right)^{\beta/p} - \frac{1}{2}\frac{1}{K^2}\sum_{k=1}^{K}\sum_{l=1}^{K}\left(\sum_{t=1}^{T}\sum_{s=1}^{S}|x_k^{t,s} - \tilde{x}_l^{t,s}|^p\right)^{\beta/p}$$

akin to Frobenius distance for $p = 2$.

Column-wise

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$



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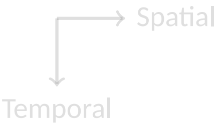
$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$

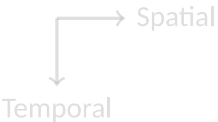


$$EST(\mathbf{F}_{\mathbf{X}_i}, \mathbf{y}_i) = \mathbb{E}_{k,s}\left(\|\mathbf{X} - \mathbf{y}\|_p^\beta\right) - \frac{1}{2}\mathbb{E}_{k,s}\left(\|\mathbf{X} - \tilde{\mathbf{X}}\|_p^\beta\right)$$

akin to Minkowski column distance.

Column-wise

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$



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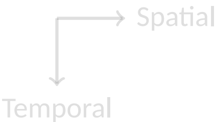
$$\widehat{EST}(\mathbf{x}_i, \mathbf{y}_i) = \frac{1}{K}\sum_{k=1}^{K}\frac{1}{S}\sum_{s=1}^{S}\left(\sum_{t=1}^{T}|x_k^{t,s} - y^{t,s}|^p\right)^{\beta/p} - \frac{1}{2K^2}\sum_{k=1}^{K}\sum_{l=1}^{K}\frac{1}{S}\sum_{s=1}^{S}\left(\sum_{t=1}^{T}|x_k^{t,s} - \tilde{x}_l^{t,s}|^p\right)^{\beta/p}$$

akin to Minkowski column distance.

Column-wise

2 Methodology

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$



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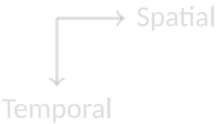
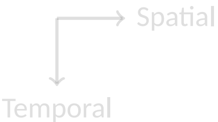
$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$



Row-wise

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$



$$ESS(\mathbf{F}_{\mathbf{X}_i}, \mathbf{y}_i) = \mathbb{E}_{k,t}\left(\|\mathbf{X} - \mathbf{y}\|_p^\beta\right) - \frac{1}{2}\mathbb{E}_{k,t}\left(\|\mathbf{X} - \tilde{\mathbf{X}}\|_p^\beta\right)$$

akin to Minkowski row distance.

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$

Spatial

Temporal

$$ESS(\mathbf{F}_{\mathbf{X}_i}, \mathbf{y}_i) = \mathbb{E}_{k,t}\left(\|\mathbf{X} - \mathbf{y}\|_p^\beta\right) - \frac{1}{2}\mathbb{E}_{k,t}\left(\|\mathbf{X} - \tilde{\mathbf{X}}\|_p^\beta\right)$$

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akin to Minkowski row distance.

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \dots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \dots & x_k^{TS} \end{bmatrix} \qquad \mathbf{y}_i = \begin{bmatrix} y^{11} & \dots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \dots & y^{TS} \end{bmatrix}$$

Spatial

Temporal

$$ESS(\mathbf{F}_{\mathbf{X}_i}, \mathbf{y}_i) = \mathbb{E}_{k,t}\left(\|\mathbf{X} - \mathbf{y}\|_p^\beta\right) - \frac{1}{2}\mathbb{E}_{k,t}\left(\|\mathbf{X} - \tilde{\mathbf{X}}\|_p^\beta\right)$$

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akin to Minkowski row distance.

The three variations

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \cdots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \cdots & x_k^{TS} \end{bmatrix} \quad \mathbf{y}_i = \begin{bmatrix} y^{11} & \cdots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \cdots & y^{TS} \end{bmatrix}$$

ES Spatio-temporal (ES)

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \cdots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \cdots & x_k^{TS} \end{bmatrix} \quad \mathbf{y}_i = \begin{bmatrix} y^{11} & \cdots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \cdots & y^{TS} \end{bmatrix}$$

ES Temporal (EST)

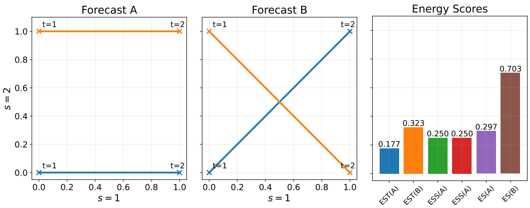
Averaged over the **spatial** dimension!

$$\mathbf{x}_{i,k} = \begin{bmatrix} x_k^{11} & \cdots & x_k^{1S} \\ \vdots & \ddots & \vdots \\ x_k^{T1} & \cdots & x_k^{TS} \end{bmatrix} \quad \mathbf{y}_i = \begin{bmatrix} y^{11} & \cdots & y^{1S} \\ \vdots & \ddots & \vdots \\ y^{T1} & \cdots & y^{TS} \end{bmatrix}$$

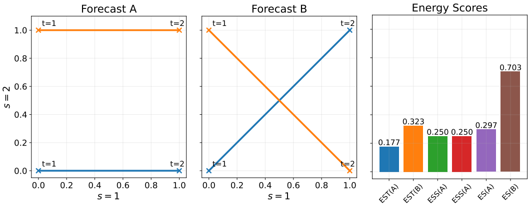
ES Spatial (ESS)

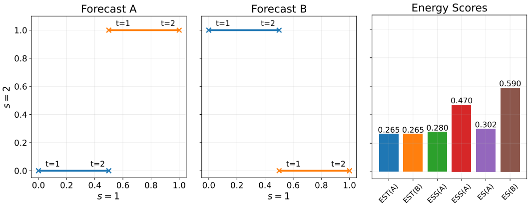
Averaged over the **temporal** dimension!

Toy Example Demonstration



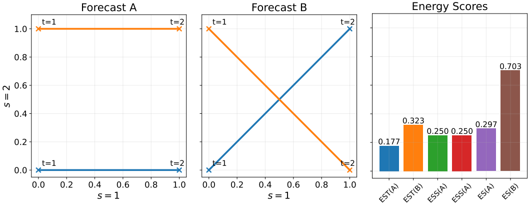
Toy Example Demonstration

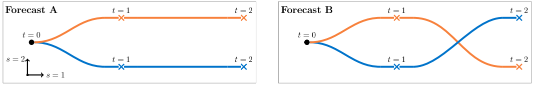


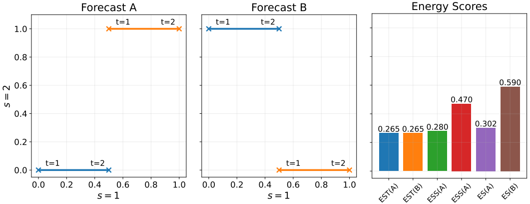


Toy Example Demonstration

2 Methodology







Toy Example Demonstration

2 Methodology

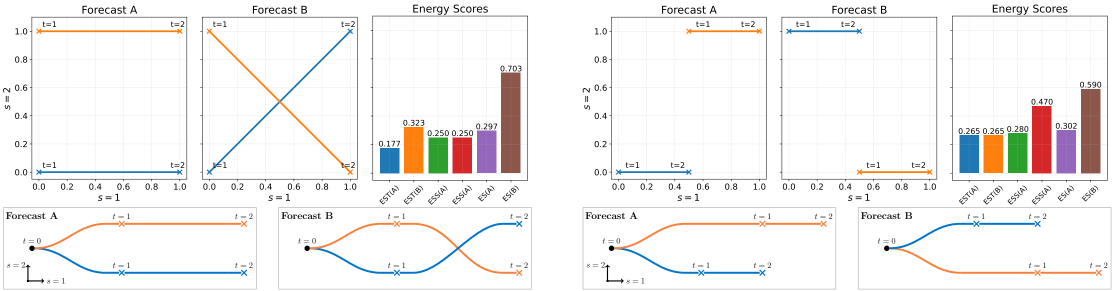


Table of Contents

▶ Introduction

▶ Methodology

▶ Experiments and Results

▶ Conclusion

Autoregressive Process

$$y_{i,k}^{t} = y_{i,k}^{t-1} + \mathcal{N}(\mu^{t} + a^{t}, (\sigma^{t} + b^{t})^{2})$$

where $t \in \{1, 2, 3\}, \quad i = [0, N), \quad k = [0, K), \quad y^{0} = 0$

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- $\mu^{t} = 1$, $\sigma^{t} = 0.2$, $a^{t} = 0$, and $b^{t} = 0$ for $t = \{1, 2, 3\}$.
- Generate $N = 5000$ observations and consider $K = \{10, 20, 50, 100, 300\}$.

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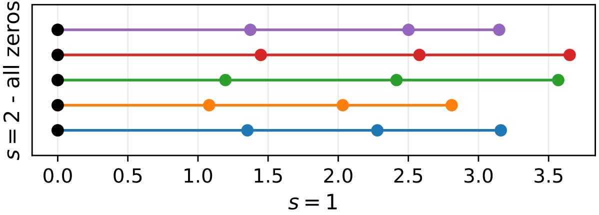
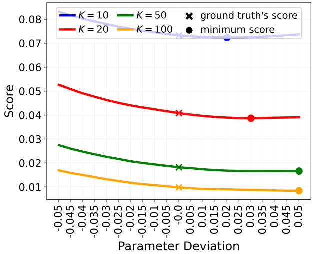
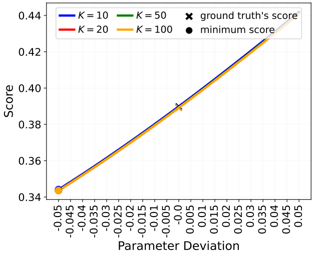
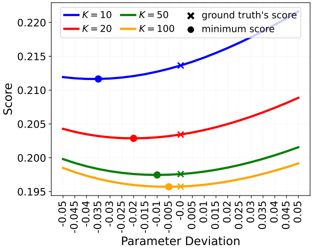




$FDE_{(L=1)}$



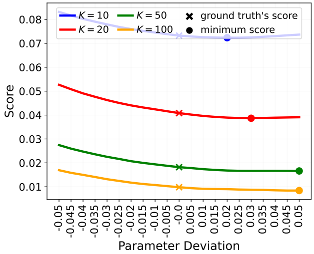
$FDE_{(L=K)}$



FES

- X-axis: predictions with different deviations.

Propriety Showcase



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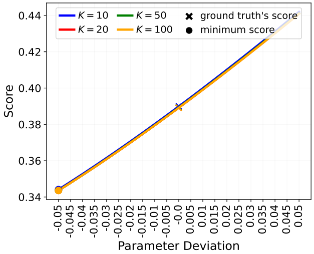
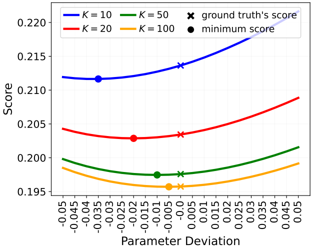


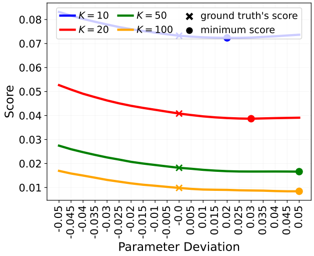
$FDE_{(L=K)}$



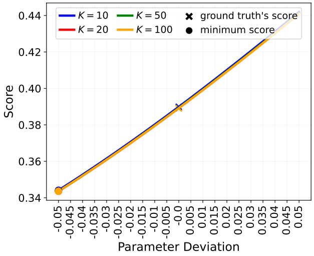
FES

- X-axis: predictions with different deviations.
- A strictly proper metric gets minimized at the optimal parameter ($deviation = 0$).

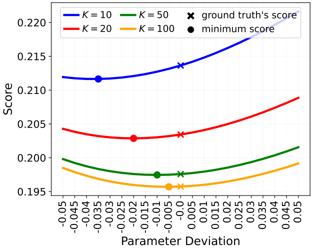
Propriety Showcase



$FDE_{(L=1)}$

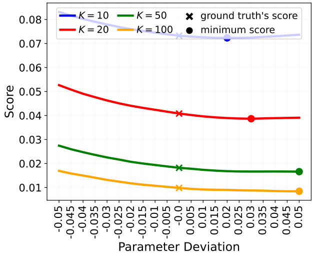


$FDE_{(L=K)}$



FES

- X-axis: predictions with different deviations.
- A strictly proper metric gets minimized at the optimal parameter ($deviation = 0$).
- As a reminder: $FDE_{(L=1)} \equiv minFDE$ and $FDE_{(L=K)} \equiv FDE$.



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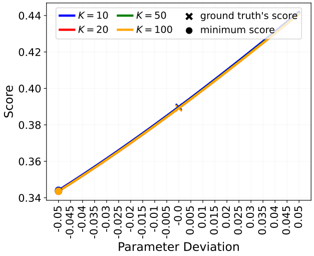
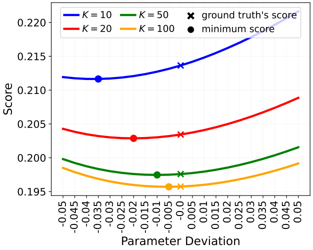


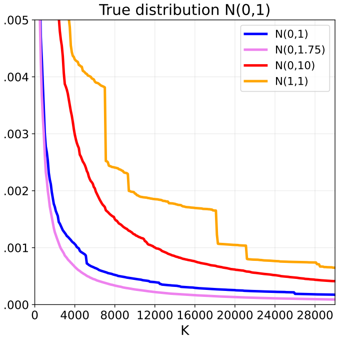
$FDE_{(L=K)}$



FES

Results support Prepositions 4.1 and 4.2

Effect of Sample Size



$$FDE_{(L=1)}$$

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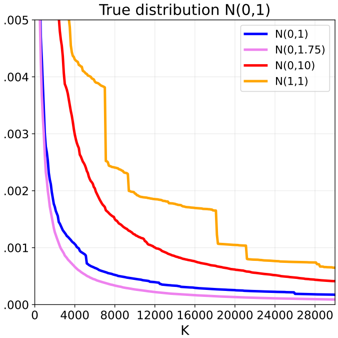

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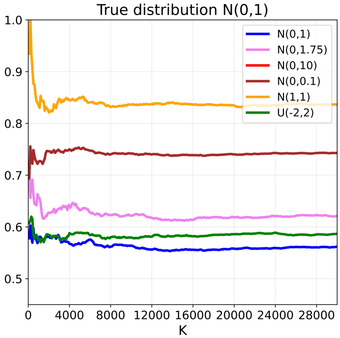

FES

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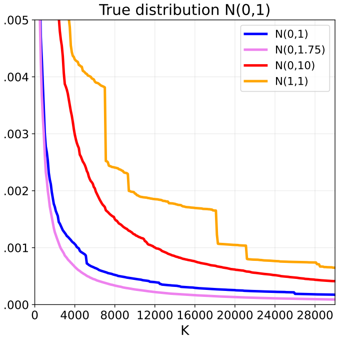

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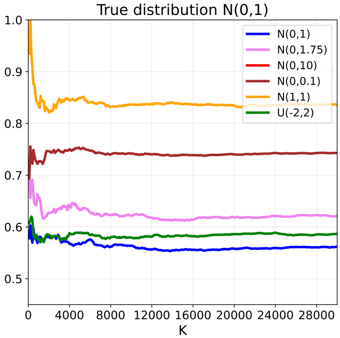

FES

Results support Proposition 4.4

ETH/UCY Dataset

- Pretrained models from Bae et. al [5] on ETH/UCY human trajectory datasets.

[5] Bae, I., Park, J. H., & Jeon, H. G. (2022). Non-probability sampling network for stochastic human trajectory prediction. In Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition (pp. 6477-6487).

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- **Goal:** how Energy Score ranks differently than its MoN counterpart?

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ETH/UCY Dataset

Reported values: expected $minADE/ES$. AVG is the arithmetic average over all datasets. **Bold**: best model, underline: second best model. Baselines: *-MC.

	ETH	HOTEL	UNIV	ZARA1	ZARA2	AVG
stgcnn-mc	0.65/1.44	0.50/1.05	0.44/0.96	0.34/0.77	0.30/0.67	0.45/0.98
pecnet-mc	0.61/1.64	0.22/0.70	0.33/0.89	0.25/0.74	0.19/0.65	0.32/0.92
sgcn-mc	0.57/1.34	0.31/0.73	0.37/0.85	0.29/0.68	0.22/0.53	0.35/<u>0.82</u>
stgcnn-qmc	0.61/<u>1.30</u>	0.34/0.98	0.36/0.89	0.32/0.74	0.29/0.65	0.38/0.91
pecnet-qmc	0.60/1.62	0.21/0.68	0.33/0.88	0.24/0.72	0.18/0.62	0.31/0.91
sgcn-qmc	0.49/**1.23**	0.21/0.66	0.31/**0.78**	0.25/**0.63**	0.19/**0.49**	0.29/**0.76**
stgcnn-npsn	<u>0.44</u>/1.48	0.21/0.88	<u>0.28</u>/0.88	0.25/0.83	0.22/0.73	<u>0.28</u>/0.96
pecnet-npsn	0.55/1.60	<u>0.19</u>/<u>0.63</u>	0.29/0.88	<u>0.21</u>/0.70	<u>0.16</u>/0.56	<u>0.28</u>/0.87
sgcn-npsn	**0.36/1.23**	**0.16/0.62**	**0.23**/<u>0.79</u>	**0.18**/<u>0.66</u>	**0.14**/<u>0.50</u>	**0.21/0.76**

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Reported values: expected $minADE/ES$. AVG is the arithmetic average over all datasets. **Bold**: best model, underline: second best model. Baselines: *-MC.

	ETH	HOTEL	UNIV	ZARA1	ZARA2	AVG
stgcnn-mc	0.65/1.44	0.50/1.05	0.44/0.96	0.34/0.77	0.30/0.67	0.45/0.98
pecnet-mc	0.61/1.64	0.22/0.70	0.33/0.89	0.25/0.74	0.19/0.65	0.32/0.92
sgcn-mc	0.57/1.34	0.31/0.73	0.37/0.85	0.29/0.68	0.22/0.53	0.35/0.82
stgcnn-qmc	0.61/1.30	0.34/0.98	0.36/0.89	0.32/0.74	0.29/0.65	0.38/0.91
pecnet-qmc	0.60/1.62	0.21/0.68	0.33/0.88	0.24/0.72	0.18/0.62	0.31/0.91
sgcn-qmc	0.49/**1.23**	0.21/0.66	0.31/**0.78**	0.25/**0.63**	0.19/**0.49**	0.29/**0.76**
stgcnn-npsn	0.44/1.48	0.21/0.88	0.28/0.88	0.25/0.83	0.22/0.73	0.28/0.96
pecnet-npsn	0.55/1.60	0.19/0.63	0.29/0.88	0.21/0.70	0.16/0.56	0.28/0.87
sgcn-npsn	**0.36/1.23**	**0.16/0.62**	**0.23**/0.79	**0.18**/0.66	**0.14**/0.50	**0.21/0.76**

ES favors SGCN-QMC as the best model over SGCN-NPSN on 3 out of 5 datasets, in contrast to $minADE$.

Table of Contents

- Introduction
- Methodology
- Experiments and Results
- Conclusion

Summary

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- Adopted **Scoring Rules framework** for evaluation of trajectory distribution predictions.
- Examined $minADE/minFDE$ as common instances of the **Minimum of N** family of evaluation metrics and showed that they are **improper**.
- Proposed **Energy Score-based evaluation** as an alternative.
- Demonstrated how **marginalized variations of Energy Score** can be useful for diagnosis of trajectory distribution predictions.

Limitations of Energy Score

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- Energy Score calculation has $\mathcal{O}(K^2)$ computational complexity.
- Behavior of energy score for small values of K merits further investigation.

Evaluation of Trajectory Distribution Predictions with Energy Score

Thank you for listening!

Any questions?