



西安交通大学  
XI'AN JIAOTONG UNIVERSITY

**IAIR** Est.  
1986

Institute of  
Artificial Intelligence  
and Robotics

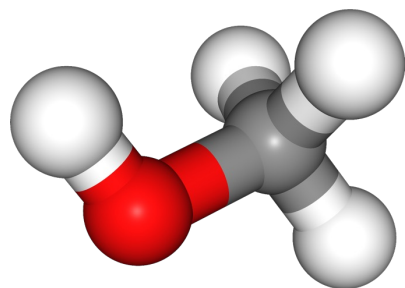


## Self-Consistency Training for Density-Functional-Theory Hamiltonian Prediction

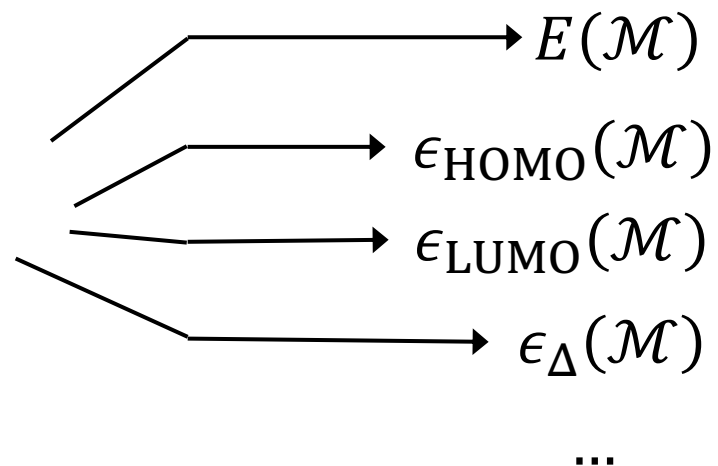
He Zhang<sup>1 2</sup>, Chang Liu<sup>2 #</sup>, Zun Wang<sup>2</sup>, Xinran Wei<sup>2</sup>, Siyuan Liu<sup>2</sup>,  
Nanning Zheng<sup>1 #</sup>, Bin Shao<sup>2</sup>, Tie-Yan Liu<sup>2</sup>

<sup>1</sup>Xi'an Jiaotong University <sup>2</sup>Microsoft AI for Science <sup>#</sup>Corresponding authors

# Hamiltonian Prediction

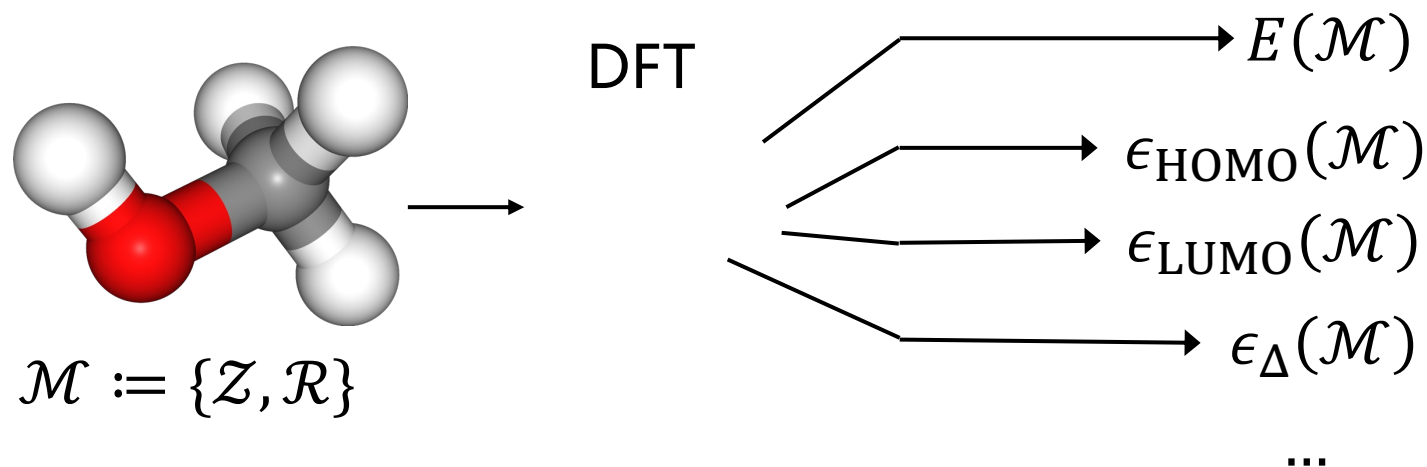


$$\mathcal{M} := \{Z, \mathcal{R}\}$$



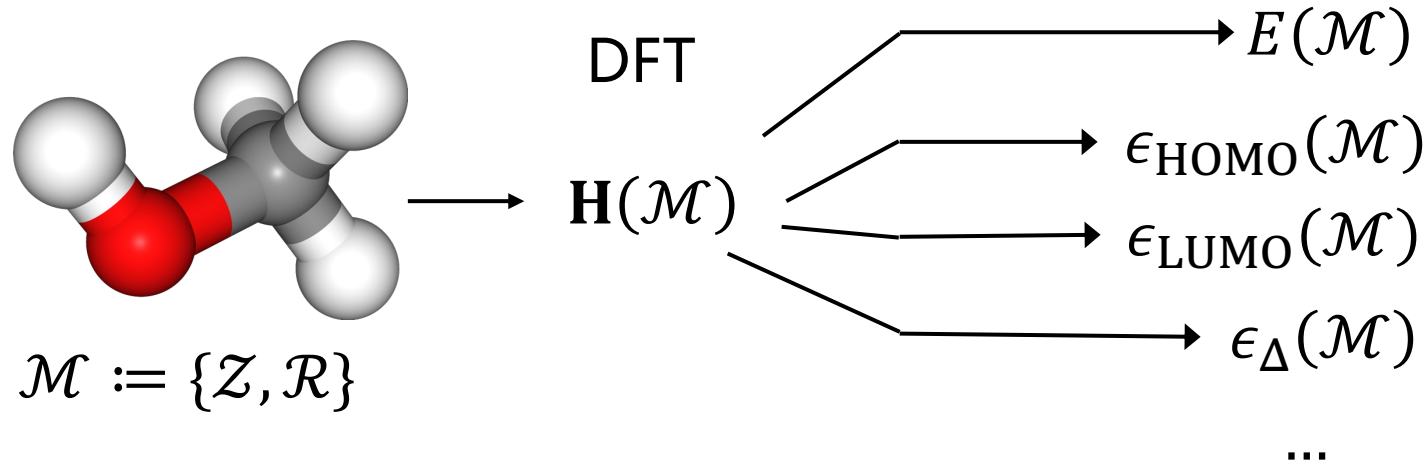
- Molecular properties: interaction among electrons and atomic nuclei

# Hamiltonian Prediction



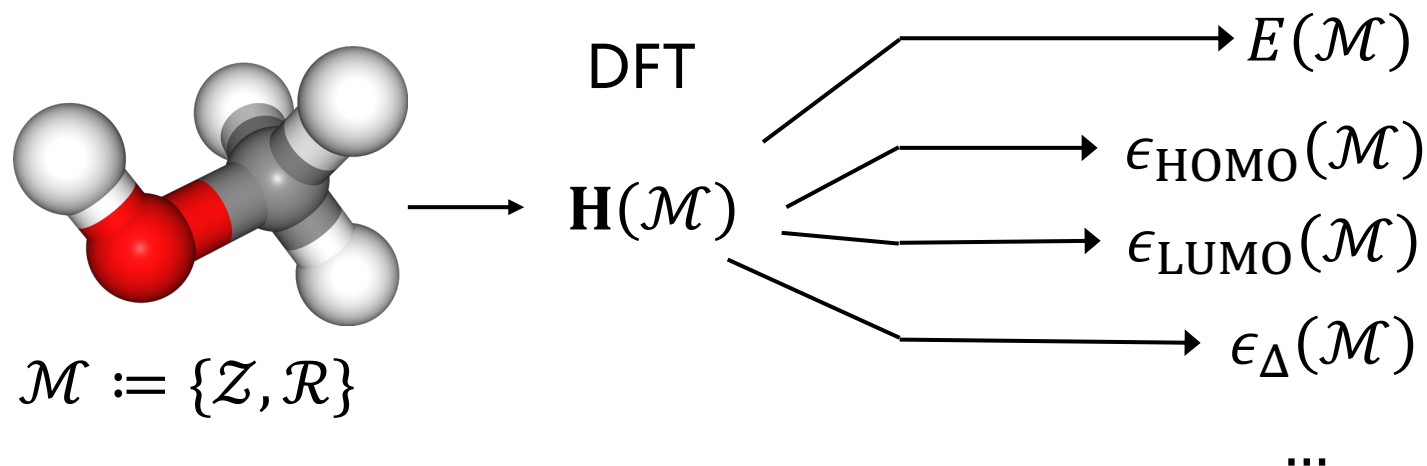
- Molecular properties: interaction among electrons and atomic nuclei
- DFT: solve electronic structure hence properties

# Hamiltonian Prediction



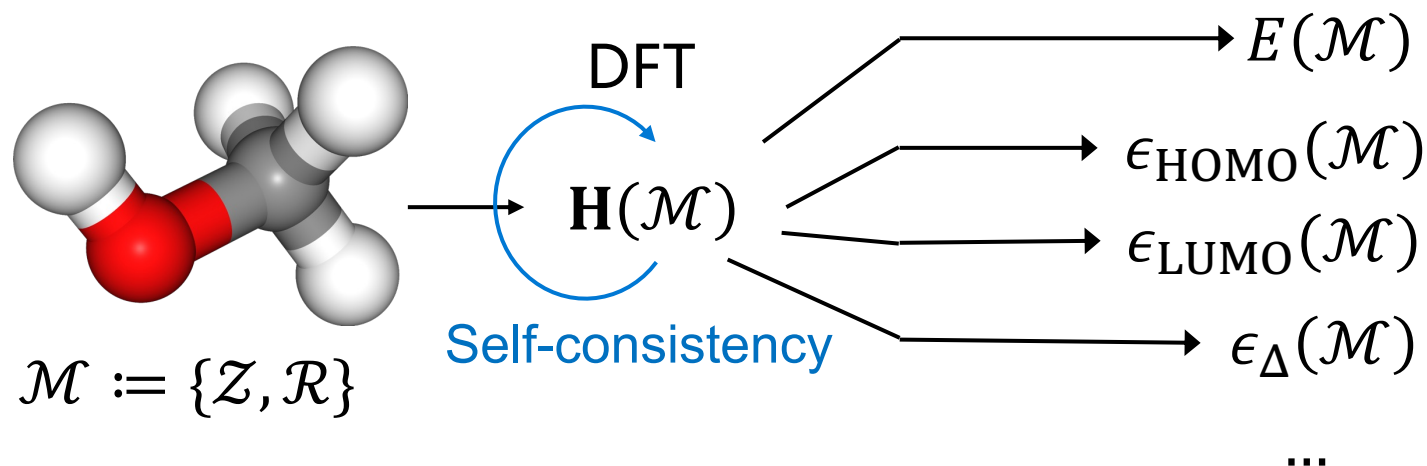
- Molecular properties: interaction among electrons and atomic nuclei
- DFT: solve electronic structure hence properties
- Hamiltonian: raw DFT solution, derive all properties

# Hamiltonian Prediction



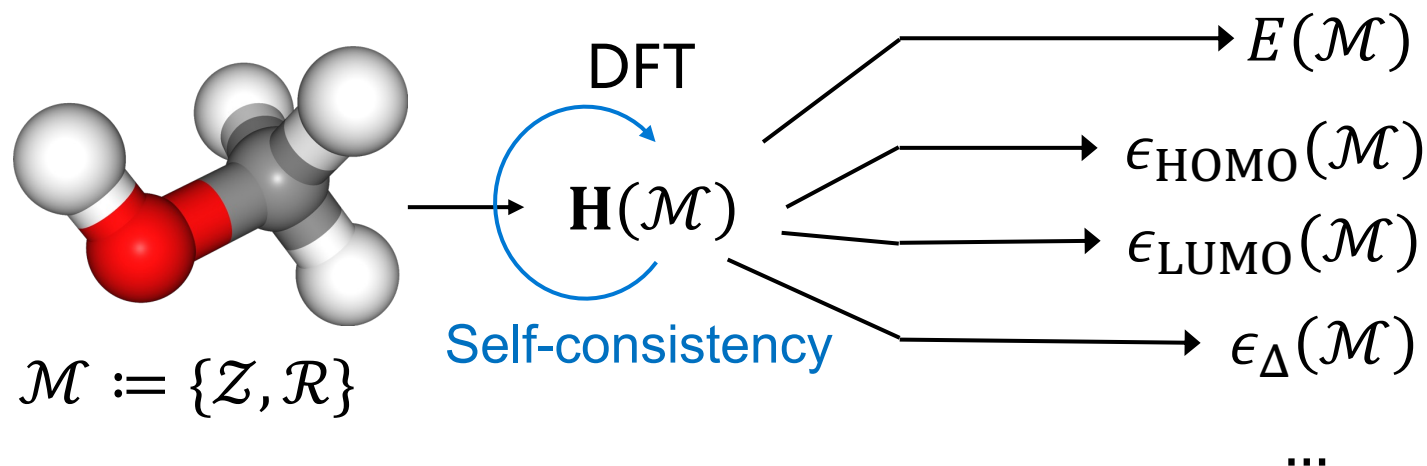
- Molecular properties: interaction among electrons and atomic nuclei
- DFT: solve electronic structure hence properties
- Hamiltonian: raw DFT solution, derive all properties
- Hamiltonian prediction:  
“unified” predictor, provide all properties that DFT can

# Hamiltonian Prediction



- Hamiltonian prediction has a **self-consistency** principle: **Training without label!**
  - Distinction from common property prediction: data-free training / self-improvement
  - Compensating **data scarcity** with scientific laws

# Hamiltonian Prediction



- Hamiltonian prediction has a **self-consistency** principle: **Training without label!**
  - **Distinction** from common property prediction: data-free training / self-improvement
  - Compensating **data scarcity** with **scientific laws**
- Unique benefits:
  - Exact **generalization** to arbitrary workload beyond labeled data
  - Amortization of DFT calculation: **more efficient** than running DFT to generate labels



# Background: DFT Formulation

- Describe  $N$ -electron state by orbitals  $\{\phi_i(\mathbf{r})\}_{i=1}^N \rightarrow$  coefficients  $\mathbf{C}$  under a basis set
- Solve for the electron state  $\mathbf{C}$  of molecular structure  $\mathcal{M}$  by minimizing:

$$E_{\mathcal{M}}(\mathbf{C}), \text{ s.t. } \mathbf{C}^T \mathbf{S}_{\mathcal{M}} \mathbf{C} = \mathbf{I}.$$

- Solve the optimization problem:

$$\underbrace{\mathbf{H}_{\mathcal{M}}(\mathbf{C})}_{\text{Kohn-sham equation}} \mathbf{C} = \mathbf{S}_{\mathcal{M}} \mathbf{C} \epsilon.$$

$$:= \frac{1}{2} \nabla_{\mathbf{C}} E_{\mathcal{M}}(\cdot)$$

Kohn-sham equation





# Background: DFT Formulation

- Describe  $N$ -electron state by orbitals  $\{\phi_i(\mathbf{r})\}_{i=1}^N \rightarrow$  coefficients  $\mathbf{C}$  under a basis set
- Solve for the electron state  $\mathbf{C}$  of molecular structure  $\mathcal{M}$  by minimizing:

$$E_{\mathcal{M}}(\mathbf{C}), \text{ s.t. } \mathbf{C}^T \mathbf{S}_{\mathcal{M}} \mathbf{C} = \mathbf{I}.$$

- Solve the optimization problem:

$$\mathbf{H}_{\mathcal{M}}(\mathbf{C}) \mathbf{C} = \mathbf{S}_{\mathcal{M}} \mathbf{C} \epsilon.$$

$$\underbrace{\quad}_{:= \frac{1}{2} \nabla_{\mathbf{C}} E_{\mathcal{M}}(\cdot)}$$

Kohn-sham equation



Self-Consistent Field (SCF) iteration:

$$\mathbf{C}^{(k-1)} \rightarrow \mathbf{H}^{(k)} = \mathbf{H}_{\mathcal{M}}(\mathbf{C}^{(k-1)}) \rightarrow \mathbf{C}^{(k)} := \mathbf{C}_{\mathcal{M}}(\mathbf{H}^{(k)}) \text{ which solves } \mathbf{H}^{(k)} \mathbf{C} = \mathbf{S}_{\mathcal{M}} \mathbf{C} \epsilon$$

$\rightarrow \mathbf{H}_{\mathcal{M}}^*$  after convergence

# DFT Calculation $\rightarrow$ Self-Consistency Training



Kohn-Sham equation

$$\mathbf{H}_{\mathcal{M}}(\mathbf{C}) \mathbf{C} = \mathbf{S}_{\mathcal{M}} \mathbf{C} \epsilon$$



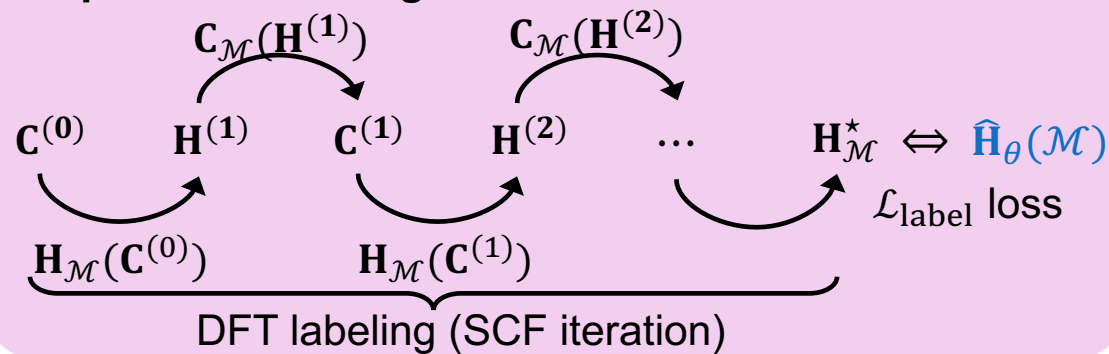
$\mathbf{H}$

$\mathbf{C}$

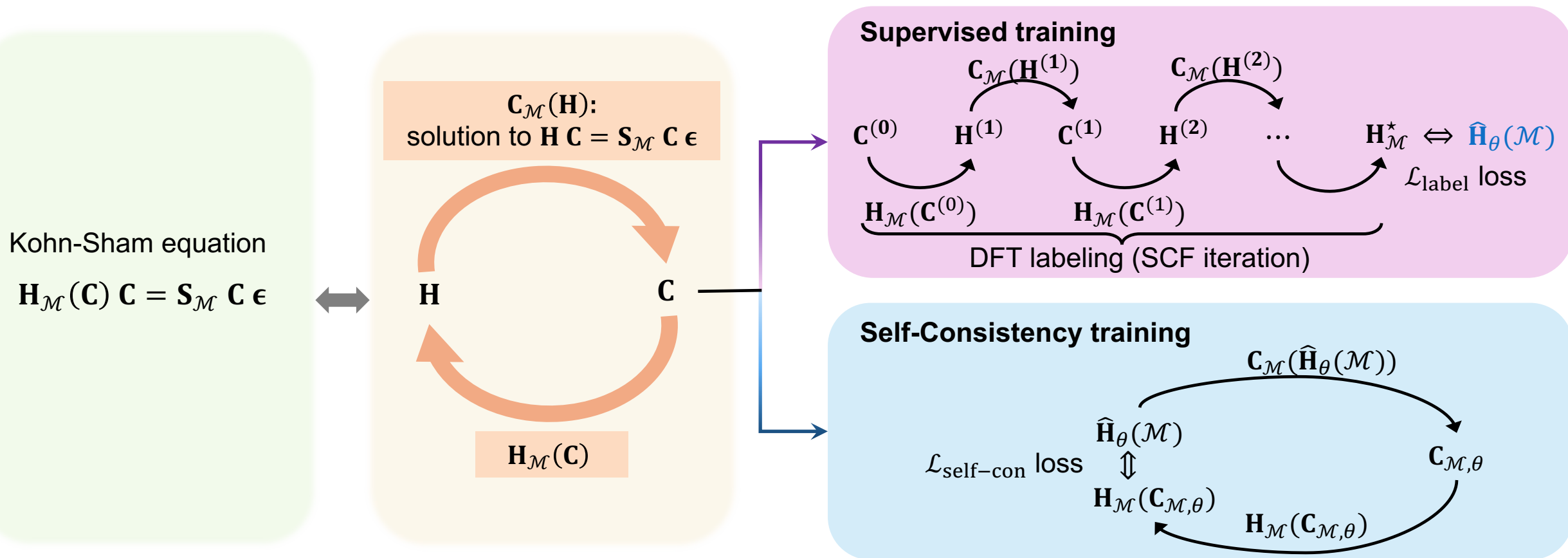
$\mathbf{H}_{\mathcal{M}}(\mathbf{C})$

$\mathbf{C}_{\mathcal{M}}(\mathbf{H})$ :  
solution to  $\mathbf{H} \mathbf{C} = \mathbf{S}_{\mathcal{M}} \mathbf{C} \epsilon$

Supervised training



# DFT Calculation → Self-Consistency Training



$$L_{\text{sc}}(\theta; \mathcal{D}) = \frac{1}{|\mathcal{D}|} \sum_{\mathcal{M} \sim \mathcal{D}} \left\| \hat{\mathbf{H}}_{\theta}(\mathcal{M}) - \mathbf{H}_{\mathcal{M}} \left( \mathbf{C}_{\mathcal{M}} \left( \hat{\mathbf{H}}_{\theta}(\mathcal{M}) \right) \right) \right\|_{\text{F}}^2$$

# Self-Consistency Training

- Self-consistency loss

$$L_{sc}(\theta; \mathcal{D}) = \frac{1}{|\mathcal{D}|} \sum_{\mathcal{M} \sim \mathcal{D}} \left\| \hat{\mathbf{H}}_{\theta}(\mathcal{M}) - \mathbf{H}_{\mathcal{M}} \left( \mathbf{C}_{\mathcal{M}} \left( \hat{\mathbf{H}}_{\theta}(\mathcal{M}) \right) \right) \right\|_{\text{F}}^2$$

- Not just a regularization: it determines the DFT solution (label).

# Self-Consistency Training



- Self-consistency loss

$$L_{sc}(\theta; \mathcal{D}) = \frac{1}{|\mathcal{D}|} \sum_{\mathcal{M} \sim \mathcal{D}} \left\| \hat{\mathbf{H}}_{\theta}(\mathcal{M}) - \mathbf{H}_{\mathcal{M}} \left( \mathbf{C}_{\mathcal{M}} \left( \hat{\mathbf{H}}_{\theta}(\mathcal{M}) \right) \right) \right\|_{\text{F}}^2$$

- Not just a regularization: it determines the DFT solution (label).
- Minimizing the gap unnecessarily drives  $\hat{\mathbf{H}}_{\theta}(\mathcal{M})$  towards  $\mathbf{H}_{\mathcal{M}} \left( \mathbf{C}_{\mathcal{M}} \left( \hat{\mathbf{H}}_{\theta}(\mathcal{M}) \right) \right)$ .
  - The latter may even be farther from the solution, in which case both are driven to the solution.
  - Should not apply stop-gradient to the latter.

# Self-Consistency Training



- Hamiltonian prediction
  - Roto-translational/ $SE(3)$  equivariance
  - QHNet [Yu'23]: an  $SE(3)$ -equivariant GNN balance efficiency and accuracy

# Self-Consistency Training



- Hamiltonian prediction
  - Roto-translational/SE(3) equivariance
  - QHNet [Yu'23]: an SE(3)-equivariant GNN balance efficiency and accuracy

- Hamiltonian reconstruction
$$L_{sc}(\theta; \mathcal{D}) = \frac{1}{|\mathcal{D}|} \sum_{\mathcal{M} \sim \mathcal{D}} \left\| \hat{\mathbf{H}}_{\theta}(\mathcal{M}) - \mathbf{H}_{\mathcal{M}} \left( \mathbf{C}_{\mathcal{M}} \left( \hat{\mathbf{H}}_{\theta}(\mathcal{M}) \right) \right) \right\|_{\text{F}}^2$$
  - Numerically stable implementation of differentiation through eigensolver  $\mathbf{C}_{\mathcal{M}}(\mathbf{H})$  .
  - GPU implementation of Hamiltonian construction  $\mathbf{H}_{\mathcal{M}}(\mathbf{C})$

# Unique Benefits



- Generalization beyond labeled data:  $\underbrace{\mathcal{L}_{\text{label}}(\theta; \overline{\mathcal{D}^{(1)}})}_{\text{limited labeled dataset}} + \lambda \underbrace{\mathcal{L}_{\text{self-con}}(\theta; \mathcal{D}^{(2)})}_{\text{unlimited unlabeled dataset}}$





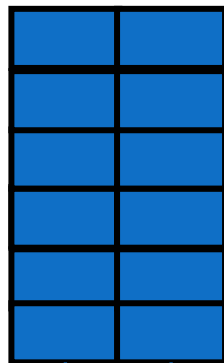
# Unique Benefits

- Generalization beyond labeled data:  $\underbrace{\mathcal{L}_{\text{label}}(\theta; \overline{\mathcal{D}^{(1)}})}_{\text{limited labeled dataset}} + \lambda \underbrace{\mathcal{L}_{\text{self-con}}(\theta; \mathcal{D}^{(2)})}_{\text{unlimited unlabeled dataset}}$
- Amortization effect: efficiency over DFT labeling

Cost of one iteration:



DFT calculation:



supervision

Training molecules:

$\mathcal{M}^{(1)} \mathcal{M}^{(2)} \mathcal{M}^{(3)} \mathcal{M}^{(4)} \mathcal{M}^{(5)} \mathcal{M}^{(6)} \mathcal{M}^{(7)} \mathcal{M}^{(8)} \mathcal{M}^{(9)} \mathcal{M}^{(10)} \mathcal{M}^{(11)} \mathcal{M}^{(12)}$



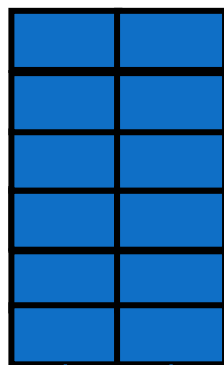
# Unique Benefits

- Generalization beyond labeled data:  $\underbrace{\mathcal{L}_{\text{label}}(\theta; \overline{\mathcal{D}^{(1)}})}_{\text{limited labeled dataset}} + \lambda \underbrace{\mathcal{L}_{\text{self-con}}(\theta; \mathcal{D}^{(2)})}_{\text{unlimited unlabeled dataset}}$
- Amortization effect: efficiency over DFT labeling

Cost of one iteration:



DFT calculation:



supervision

Training molecules:

$\mathcal{M}^{(1)} \mathcal{M}^{(2)} \mathcal{M}^{(3)} \mathcal{M}^{(4)} \mathcal{M}^{(5)} \mathcal{M}^{(6)} \mathcal{M}^{(7)} \mathcal{M}^{(8)} \mathcal{M}^{(9)} \mathcal{M}^{(10)} \mathcal{M}^{(11)} \mathcal{M}^{(12)}$

Self-consistency training:



supervision



# Results: Generalization beyond Labeled Data



- Out-of-distribution (OOD) scenario (QH9)

labeled small molecules + **finetune on unlabeled large molecules** → test on large molecules

$\mathbf{H}^{(1)}$  ○  
 $\mathbf{H}^{(2)}$  ○  
 $\vdots$  ○  
 $\mathbf{H}^{(n)}$  ○

$\mathcal{D}_{\text{small}}$   
 $L_{\text{label}}$   
**pretrain**

○  
 ○  
 ○  
 ○

$L_{\text{self-con}}$   
 $\mathcal{D}_{\text{large}}$   
 $\mathcal{M}^{(d)}$

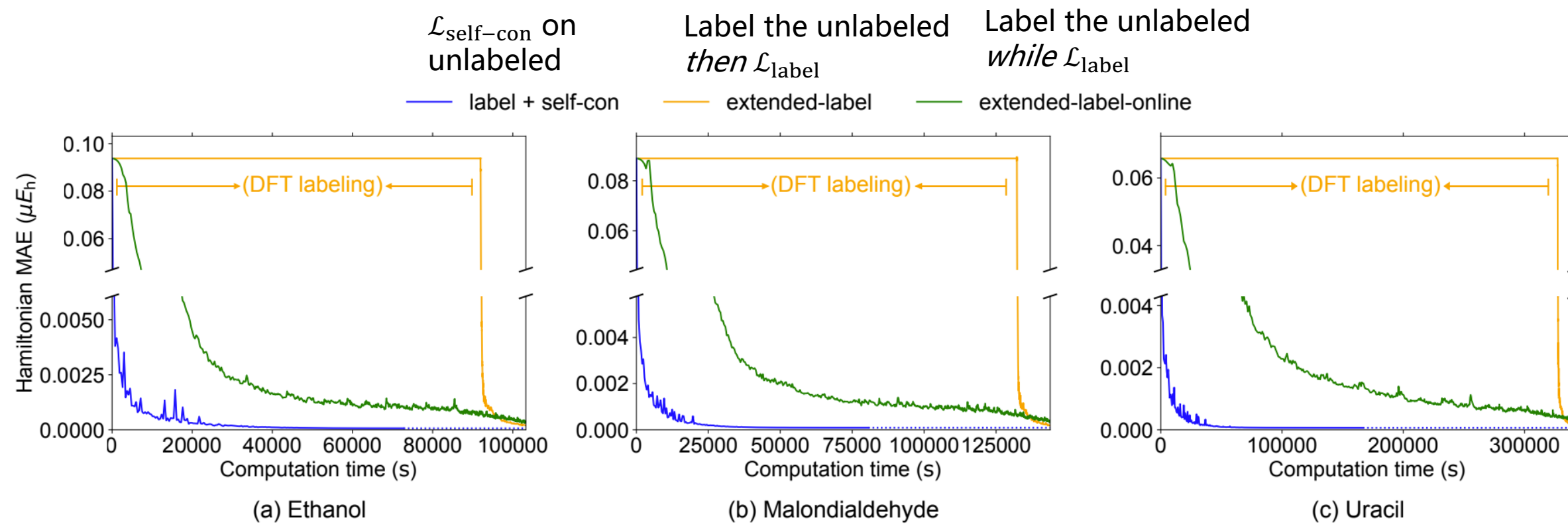
Setting	$\mathbf{H} [\mu E_h] \downarrow$	$\epsilon [\mu E_h] \downarrow$	$\mathbf{C} [\%] \uparrow$	$\epsilon_{\text{HOMO}} [\mu E_h] \downarrow$	$\epsilon_{\text{LUMO}} [\mu E_h] \downarrow$	$\epsilon_{\Delta} [\mu E_h] \downarrow$	SCF Accel. [%] $\downarrow$
zero-shot	69.67	403.52	95.72	778.86	12230.49	12203.12	66.3
self-con (all-param)	65.74	375.31	<b>97.31</b>	565.50	<b>1130.55</b>	<b>1316.96</b>	<b>64.5</b>
self-con (adapter)	<b>64.48</b>	<b>268.83</b>	97.12	<b>449.80</b>	1220.54	1394.29	65.0

**finetune**

# Results: Training Efficiency by Amortization



- Data-scarce scenario (MD17):  $|\overline{\mathcal{D}^{(1)}}| = 100, |\mathcal{D}^{(2)}| = 24900$

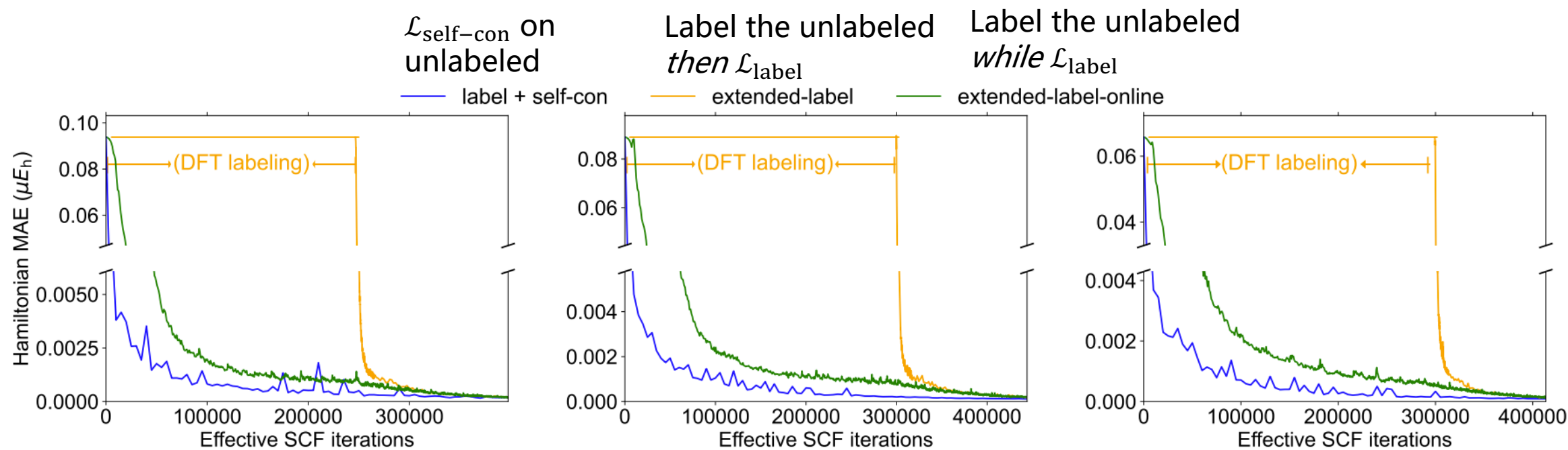


accuracy-cost (*computation time*) curve

# Results: Amortization of DFT



- Data-scarce scenario (MD17):  $|\overline{\mathcal{D}^{(1)}}| = 100, |\mathcal{D}^{(2)}| = 24900$



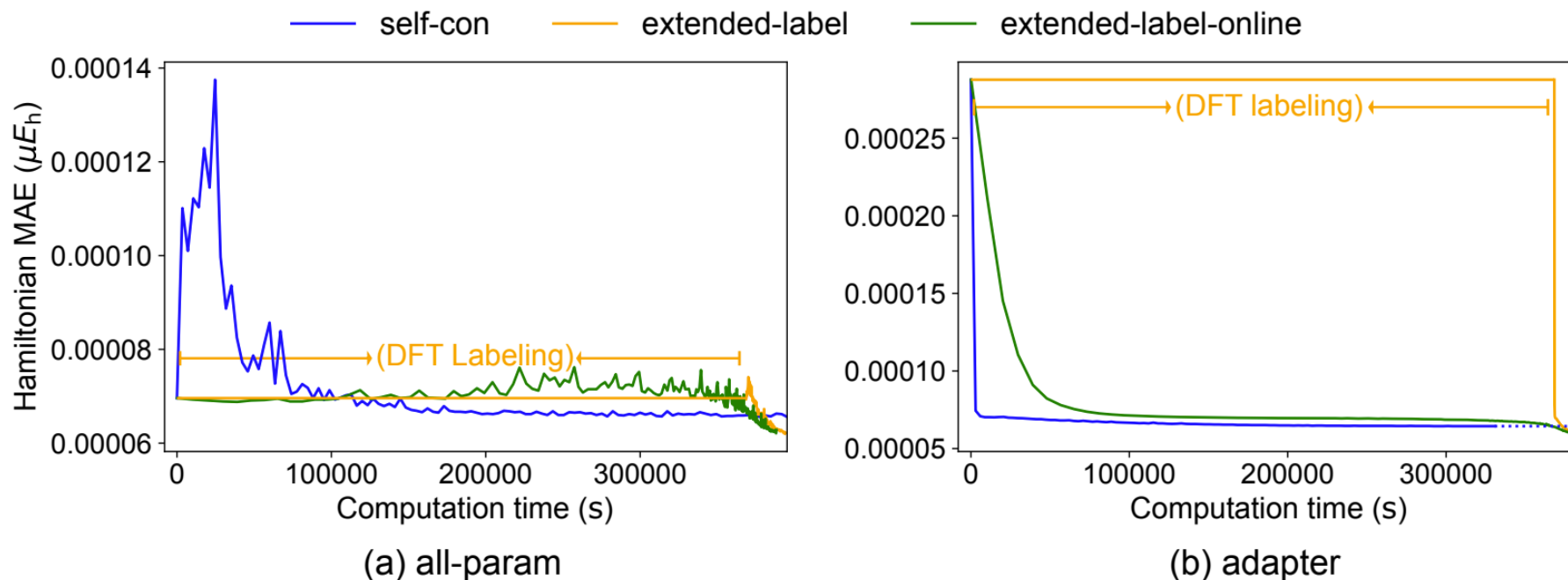
accuracy-cost (*effective SCF iterations*) curve

# Results: Amortization of DFT



- OOD scenario (QH9)

labeled small molecules + **finetune on unlabeled large molecules** → test on large molecules

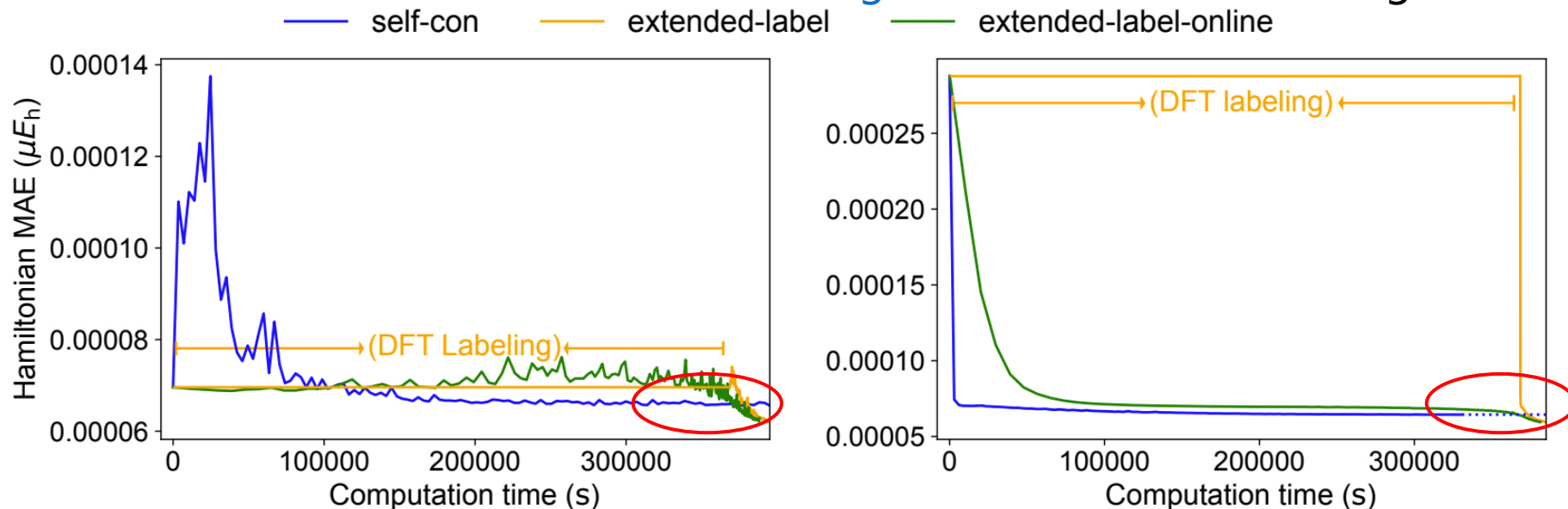


*accuracy-cost (computation time) curve*

# Results: Amortization of DFT

- OOD scenario (QH9)

labeled small molecules + **finetune on unlabeled large molecules** → test on large molecules



**Final performance:** self-consistency training gives better derived molecular properties!

FT mode	Setting	H [ $\mu E_h$ ] ↓	$\epsilon$ [ $\mu E_h$ ] ↓	C [%] ↑	$\epsilon_{HOMO}$ [ $\mu E_h$ ] ↓	$\epsilon_{LUMO}$ [ $\mu E_h$ ] ↓	$\epsilon_{\Delta}$ [ $\mu E_h$ ] ↓	SCF Accel. [%] ↓
all-param	extended-label	<b>62.13</b>	<b>365.66</b>	96.89	577.46	5962.16	6137.66	65.0
	self-con	65.74	375.31	<b>97.31</b>	<b>565.50</b>	<b>1130.55</b>	<b>1316.96</b>	<b>64.5</b>
adapter	extended-label	<b>59.67</b>	330.05	96.63	541.92	6372.12	6445.33	65.2
	self-con	64.48	<b>268.83</b>	<b>97.12</b>	<b>449.80</b>	<b>1220.54</b>	<b>1394.29</b>	<b>65.0</b>



# Results: Amortization of DFT

- Direct acceleration over DFT calculation
  - Self-consistency training time vs. DFT computation time to reach the same level of electronic energy accuracy

Molecule	criterion [ $\mu E_h$ ]	$t_{\text{self-con}}$ [s]	$t_{\text{DFT}}$ [s]
Ethanol	31.0	<b><math>4.50 \times 10^4</math></b>	$6.40 \times 10^4$
Malondialdehyde	88.9	<b><math>4.81 \times 10^4</math></b>	$1.05 \times 10^5$
Uracil	177.2	<b><math>1.23 \times 10^5</math></b>	$2.15 \times 10^5$

# Results: Extending Applicable Scale of Hamiltonian Prediction

- Labeled QM9 ( $\leq 31$  atoms) + Finetune on unlabeled larger molecules  $\rightarrow$  test on MD22 (ALA3: 42 atoms, DHA: 56 atoms)
  - vs. zero-shot generalization
  - vs. SOTA end-to-end property predictors

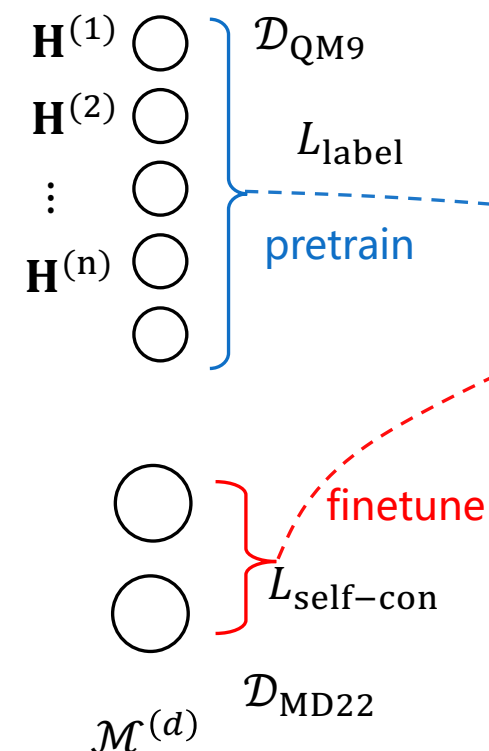


Table 5. Generalization results on large-scale molecules. All metrics are calculated on MD22 test structures.

Molecule	Setting	$\mathbf{H}$ [ $\mu E_h$ ] $\downarrow$	$\epsilon$ [ $\mu E_h$ ] $\downarrow$	$\mathbf{C}$ [%] $\uparrow$	$\epsilon_{\text{HOMO}}$ [ $\mu E_h$ ] $\downarrow$	$\epsilon_{\text{LUMO}}$ [ $\mu E_h$ ] $\downarrow$	$\epsilon_{\Delta}$ [ $\mu E_h$ ] $\downarrow$	SCF Accel. [%] $\downarrow$
ALA3	zero-shot	237.71	$6.54 \times 10^3$	52.24	$6.90 \times 10^3$	$9.51 \times 10^4$	$9.79 \times 10^4$	84.6
	self-con	<b>52.49</b>	<b><math>1.22 \times 10^3</math></b>	<b>94.46</b>	<b><math>2.07 \times 10^3</math></b>	<b><math>3.76 \times 10^3</math></b>	<b><math>2.69 \times 10^3</math></b>	<b>64.7</b>
	e2e (ET)	N/A	N/A	N/A	$1.74 \times 10^5$	$7.72 \times 10^3$	$2.38 \times 10^5$	N/A
	e2e (Equiformer)	N/A	N/A	N/A	$2.38 \times 10^5$	$1.16 \times 10^4$	$2.27 \times 10^5$	N/A
DHA	zero-shot	397.87	$1.84 \times 10^4$	20.15	$1.11 \times 10^4$	$1.90 \times 10^5$	$1.85 \times 10^5$	170.8
	self-con	<b>56.12</b>	<b><math>1.81 \times 10^3</math></b>	<b>83.51</b>	<b><math>1.99 \times 10^3</math></b>	<b><math>4.01 \times 10^3</math></b>	<b><math>2.34 \times 10^3</math></b>	<b>67.0</b>
	e2e (ET)	N/A	N/A	N/A	$2.92 \times 10^5$	$2.58 \times 10^4$	$3.39 \times 10^5$	N/A
	e2e (Equiformer)	N/A	N/A	N/A	$3.76 \times 10^5$	$2.31 \times 10^4$	$4.17 \times 10^5$	N/A

# Results: Extending Applicable Scale of Hamiltonian Prediction

- QM9 ( $\leq 31$ atoms)  $\rightarrow$  MD22 (ALA3: 42 atoms, DHA: 56 atoms)
  - vs. zero-shot generalization
  - vs. SOTA end-to-end property predictors

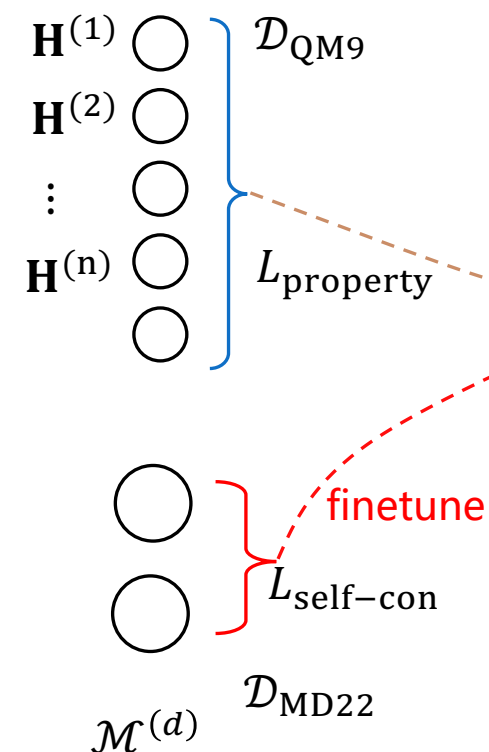


Table 5. Generalization results on large-scale molecules. All metrics are calculated on MD22 test structures.

Molecule	Setting	$\mathbf{H} [\mu E_h] \downarrow$	$\epsilon [\mu E_h] \downarrow$	$\mathbf{C} [\%] \uparrow$	$\epsilon_{\text{HOMO}} [\mu E_h] \downarrow$	$\epsilon_{\text{LUMO}} [\mu E_h] \downarrow$	$\epsilon_{\Delta} [\mu E_h] \downarrow$	SCF Accel. [%] $\downarrow$
ALA3	zero-shot	237.71	$6.54 \times 10^3$	52.24	$6.90 \times 10^3$	$9.51 \times 10^4$	$9.79 \times 10^4$	84.6
	self-con	<b>52.49</b>	<b><math>1.22 \times 10^3</math></b>	<b>94.46</b>	<b><math>2.07 \times 10^3</math></b>	<b><math>3.76 \times 10^3</math></b>	<b><math>2.69 \times 10^3</math></b>	<b>64.7</b>
	e2e (ET)	N/A	N/A	N/A	$1.74 \times 10^5$	$7.72 \times 10^3$	$2.38 \times 10^5$	N/A
	e2e (Equiformer)	N/A	N/A	N/A	$2.38 \times 10^5$	$1.16 \times 10^4$	$2.27 \times 10^5$	N/A
DHA	zero-shot	397.87	$1.84 \times 10^4$	20.15	$1.11 \times 10^4$	$1.90 \times 10^5$	$1.85 \times 10^5$	170.8
	self-con	<b>56.12</b>	<b><math>1.81 \times 10^3</math></b>	<b>83.51</b>	<b><math>1.99 \times 10^3</math></b>	<b><math>4.01 \times 10^3</math></b>	<b><math>2.34 \times 10^3</math></b>	<b>67.0</b>
	e2e (ET)	N/A	N/A	N/A	$2.92 \times 10^5$	$2.58 \times 10^4$	$3.39 \times 10^5$	N/A
	e2e (Equiformer)	N/A	N/A	N/A	$3.76 \times 10^5$	$2.31 \times 10^4$	$4.17 \times 10^5$	N/A



西安交通大学  
XI'AN JIAOTONG UNIVERSITY

**IAIR** Est.  
1986

Institute of  
Artificial Intelligence  
and Robotics



# Thank you

<https://arxiv.org/pdf/2403.09560>  
*changliu@microsoft.com*  
*mothful123@gmail.com*