Conditional Common Entropy for Instrumental Variable Testing and Partial Identification

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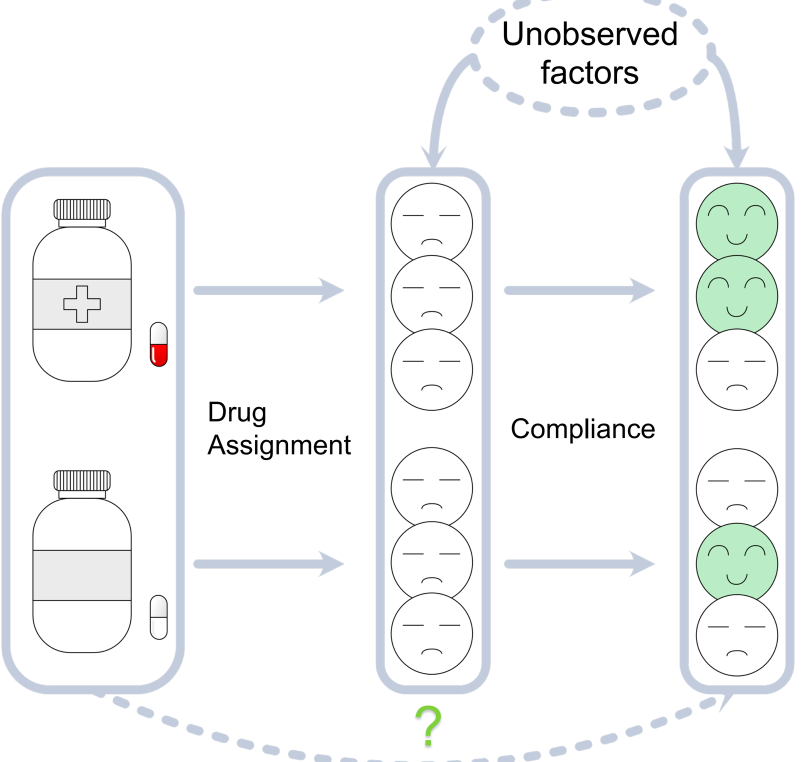
Motivation

- The instrumental variables are in general non-testable, instrumental inequality can be used to reject invalid instruments. Can we leverage the side information of latent confounder to further verify the IVs?
- How can we get tighter bounds of causal effect given the side-information about latent confounder?



Background

Instrumental Variable

a graphical representation that establishes a connection between the observational and interventional distributions.

Marginal stochastic exclusion:

$$E[Y_{z,x}] = E[Y_{z',x}]\ \forall z,z',x$$

Marginal Exchangeability:

$$Z \perp\!\!\!\perp Y_{z,x}\ \forall z,x$$

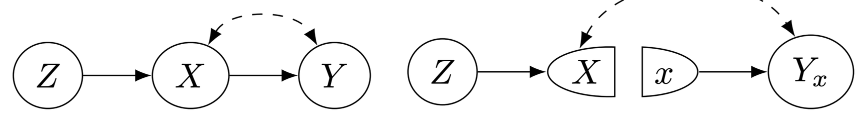
Instrumental Inequality

A necessary condition for discrete variables X, Y, Z generated from an IV graph

$$\max_x \sum_y \left[\max_z P(x,y|z)\right] \le 1$$

Single World Intervention Graph (SWIG)

a graphical representation that establishes a connection between the observational and interventional distributions with the DAG.

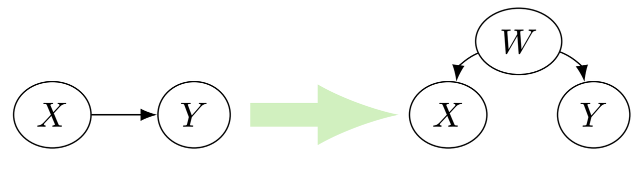


(a) IV DAG (b) Corresponding SWIG

Common Entropy

Common entropy between a pair of random variables (X, Y) is defined as the minimum entropy of a variable that make them conditionally independent.

$$CE(X;Y) \coloneqq \min_{q(x,y,w)} H(W)$$
$$s.t.\ I(X;Y|W) = 0$$
$$q(x,y,w)\ compatible\ with\ obs.$$



Conditional Common Entropy

Conditional Common Entropy

Generalize the idea of common entropy to more complex graphs.

$$CCE(Z;Y|X) \coloneqq \min_{q(x,y,z,w)} H(W)$$
$$s.t.\ I(Z;Y|X,W) = 0$$
$$q(x,y,z,w)\ compatible\ with\ obs.$$

Graph-specific Conditional Common Entropy

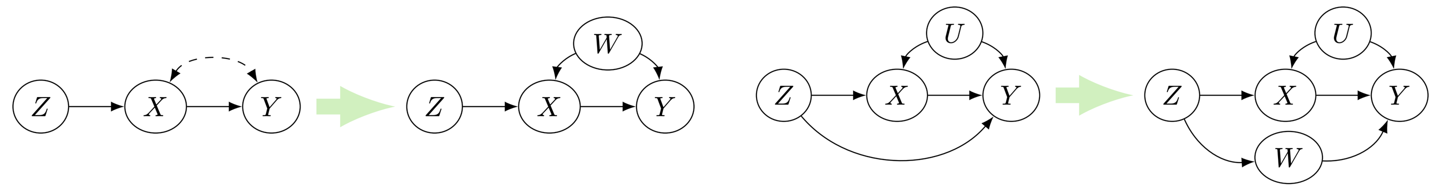
In addition to the independence constraints for CCE, we want to find a variable that attains the minimum entropy while compatible with the graph.

Definition:

Let $CCE_{v\to v'}(Z;Y|X)$ defined as the minimum entropy of the variable W, if we replace an edge $v \to v'$ with $v \to W \to v'$ and $(Z \perp\!\!\!\perp Y|X, W)$ in $\mathcal{G}'$.
Similarly define $CCE_{v\leftrightarrow v'}(Z;Y|X)$ for the double arrow edge $v \leftrightarrow v'$.

IV Graph — $H(W) = CCE_{x\leftrightarrow y}(Z;Y|X)$

Direct Graph — $H(W) = CCE_{z\to y}(Z;Y|X,U)$



$(W \perp\!\!\!\perp Z)$; $(Z \perp\!\!\!\perp Y|X,W)$ $(W \perp\!\!\!\perp U)$; $(W \perp\!\!\!\perp X|Z)$; $(Z \perp\!\!\!\perp Y|X,W,U)$

Theorem Given variables X, Y, Z in a causal graph $\mathcal{G}$ with distribution $P(X,Y,Z)$, and latent confounder U. If we have $Z \perp\!\!\!\perp Y|X,U$, then the following inequality holds

$$CCE(Z;Y|X) \le CCE_{v\leftrightarrow v'}(Z;Y|X) \le H(U)$$

Approximating Conditional Common Entropy

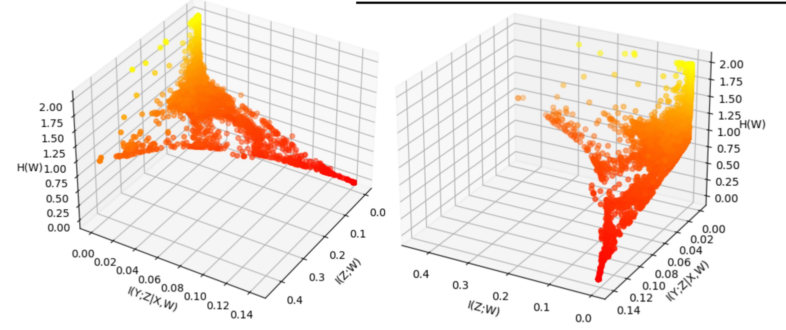
- We shows that the conditional common entropy $CCE(Z;Y|X)$ can be computed exactly for binary variables Z, Y.
- For variables with higher number of states, finding exact common entropy is NP-hard.
- We propose the **IV latent search** algorithm to approximate graph-specific conditional common entropy for higher number of states.

Relaxed loss function:

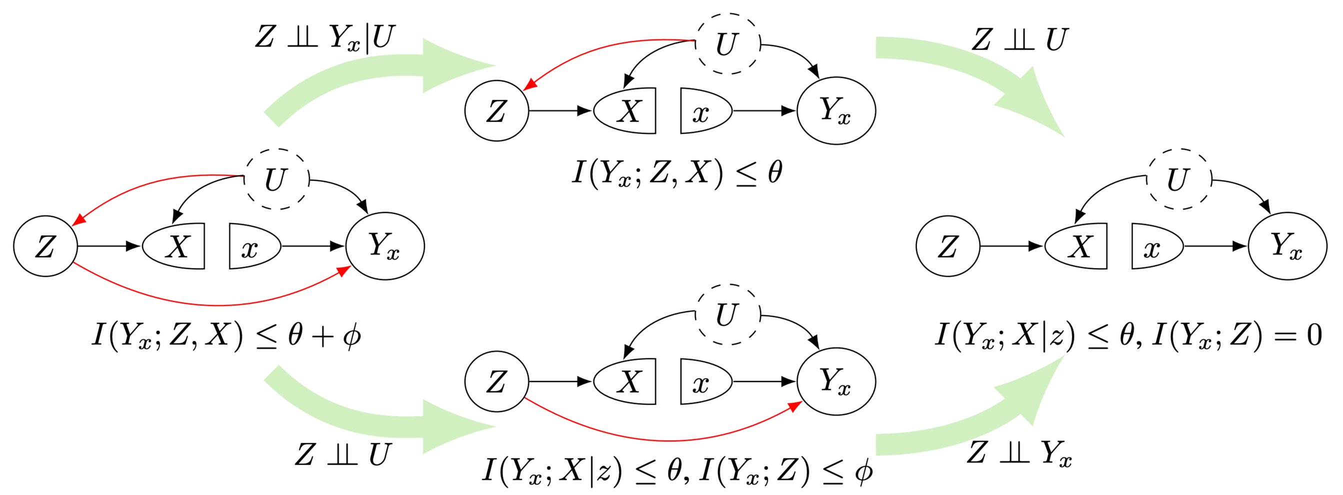
$$\mathcal{L} = I(Z;Y|X,W) + \beta_0 H(W) + \beta_1 I(W;Z)$$

Algorithm IV LatentSearch

Input: Joint distribution $P(X,Y,Z)$; Iterations number N; Initialization $q_1(W|X,Y,Z)$; $\beta_0, \beta_1 \ge 0$.
for $i \leftarrow 1\ to\ N$ **do**
  Form the joint:
   $q_i(X,Y,Z,W) \leftarrow q_i(W|X,Y,Z)P(X,Y,Z)$
  Get posteriors:
  $q_i(W) \leftarrow \sum_{x,y,z} q_i(X,Y,Z,W)$
  $q_i(W|X) \leftarrow \frac{\sum_{y,z} q_i(X,Y,Z,W)}{\sum_{y,z,w} q_i(X,Y,Z,W)}$
  $q_i(W|X,Z) \leftarrow \frac{\sum_{y} q_i(X,Y,Z,W)}{\sum_{y,w} q_i(X,Y,Z,W)}$
  $q_i(W|X,Y) \leftarrow \frac{\sum_{z} q_i(X,Y,Z,W)}{\sum_{z,w} q_i(X,Y,Z,W)}$
  Update:
  $q_{i+1}(X,Y,Z,W) \leftarrow \frac{q_i(W|X,Z)q_i(W|X,Y)q_i(U)^{\beta_0+\beta_1}}{f(X,Y,Z)q_i(W|X)q(W|Z)^{\beta_1}}$
  where $f(X,Y,Z) = \sum_u \frac{q_i(W|X,Z)q_i(W|X,Y)q_i(U)^{\beta_0+\beta_1}}{q_i(W|X)q_i(W|Z)^{\beta_1}}$
end for
Return: $q_N(W|X,Y,Z)P(X,Y,Z)$



Application of Conditional Common Entropy in IV graph



Weak Confoundedness Assumption: Consider a causal model with a set of endogenous variables $\mathcal{V}$ and exogenous variables $\mathcal{U}$. For any latent confounder $U \in \mathcal{U}$, we have $H(U) \le \theta$.

Falsify Invalid IVs

Lemma Under the weak confoundedness assumption, a covariate Z is a valid instrumental variable only if

$$CCE_{IV}(Z;Y|X) \le \theta.$$

Theorem Under Assumption 2.1, for variables X, Y, Z with $|X| = n, |Y| = m$, and $|Z| = l$ and the compatible joint distribution $P(X,Y,Z)$. Assuming $CCE_{IV}(Z;Y|X,U) = \phi$. The causal effect of x_t on y_o is bounded by

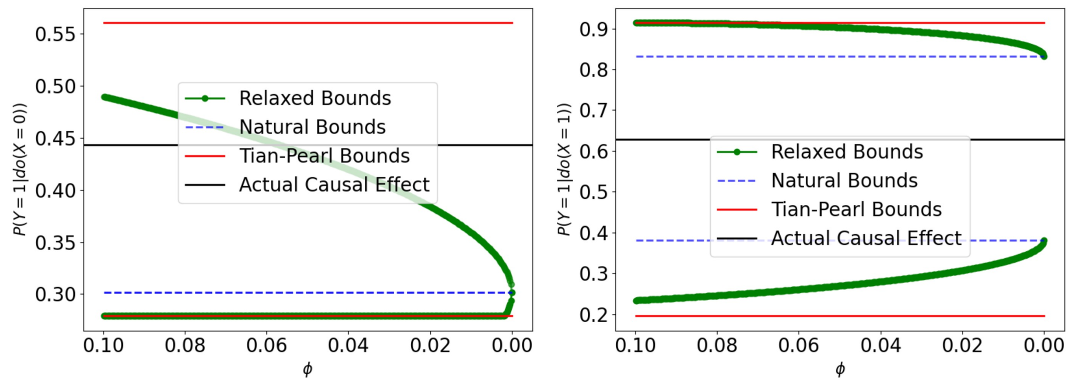
$$LB \le P(y_o|do(x_t)) \le UB,$$

where

$$LB/UB = min/\max\left(\sum_{jl} b_{ojl}P(z_l)\right)$$
$$P(z_k)b_{itk} = P(y_i, x_t, z_k)\ \forall i,k; \sum_{ij} b_{ijk} = 1\ \forall k;$$
$$\sum_i b_{ijk} = P(x_j|z_k)\ \forall j,k; 0 \le b_{ijk} \le 1\ \forall i,j,k$$
$$\sum_{ijk} b_{ijk}P(z_k)\log\left(\frac{b_{ijk}}{\left(\sum_{j'k'} b_{ij'k'}P(z_{k'})\right)\left(\sum_{i'} b_{i'jk}P(z_k)\right)}\right) \le \theta + \phi$$

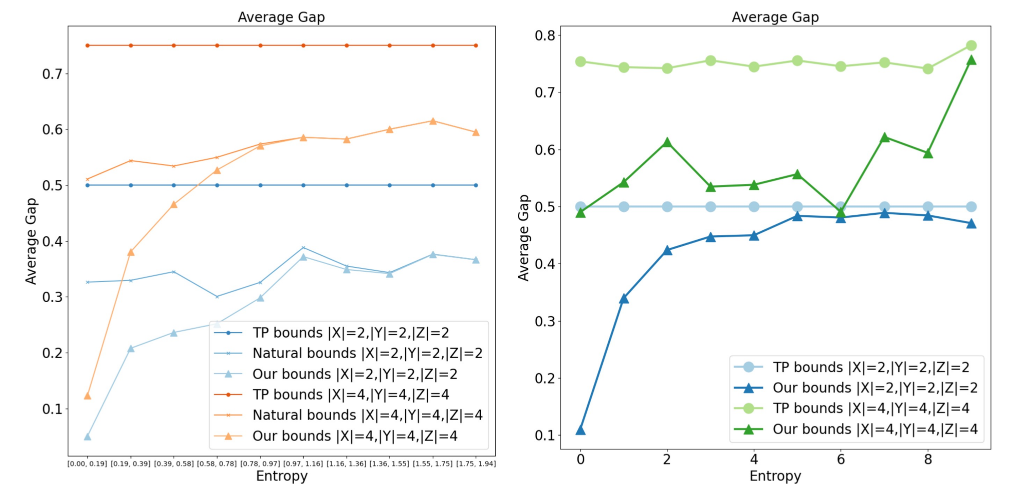
If the Marginal exchangeability assumption holds, replace the last inequality constraint with

$$\sum_{ij} b_{ijk}\log\left(\frac{b_{ijk}}{\left(\sum_{j'}(b_{ij'k})\right)\left(\sum_{i'} b_{i'jk}\right)}\right) \le \theta\ \forall k,$$
$$\sum_{ijk} b_{ijk}P(z_k)\log\left(\frac{\sum_{j'}(b_{ij'k})}{\left(\sum_{j'}(b_{ij'k})P(z_k)\right)}\right) \le \phi$$



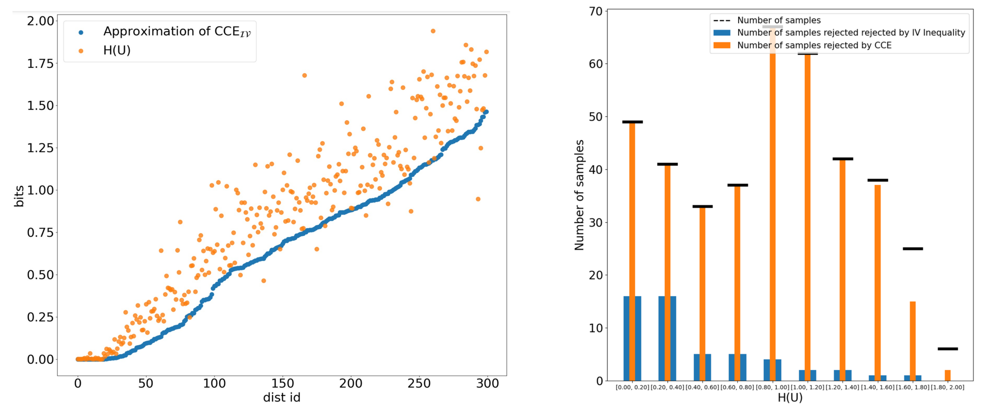
Experiment Results

Partial Identification



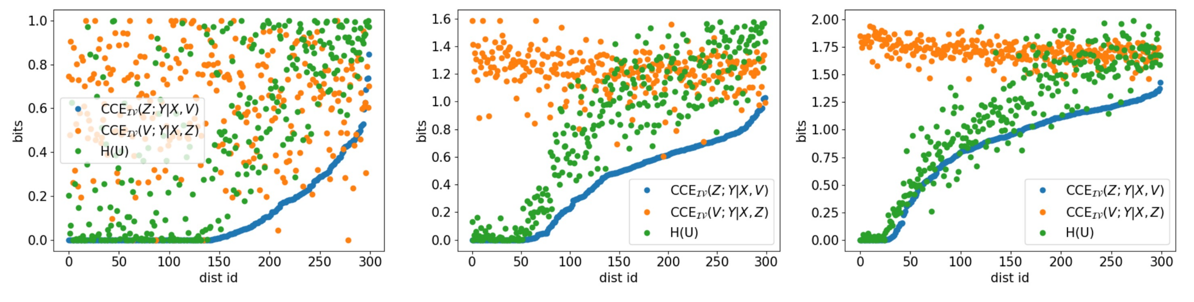
(a) IV graph $\mathcal{IV}$ (b) Invalid IV graph $\mathcal{D}$

Falsify Invalid IVs



Select IV from Covariates

- When the weak confoundedness assumption is not hold, conditional common entropy may also provide useful information for selecting IVs.
- We experimentally show that the valid IV is more likely to have smaller conditional common entropy, since it is "easier" to make it conditional independent with the outcome.



(a) $|X| = |Y| = |Z| = 2$ (b) $|X| = |Y| = |Z| = 3$ (c) $|X| = |Y| = |Z| = 4$

References

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Check out our paper and code here