Wasserstein Distributionally Robust Regret-Optimal Control Over an Infinite-Horizon

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1. Introduction

➢ **Setting:** Full-information linear time-invariant (LTI) control

- Linear plant dynamics: $\mathbf{x} = \mathcal{F}\mathbf{u} + \mathcal{G}\mathbf{w}$
- Disturbance feedback control: $\mathbf{u} = \mathcal{K}\mathbf{w}$
- Quadratic cost: $\textsc{Cost}(\mathbf{u},\mathbf{w}) := \|\mathbf{x}\|^2 + \|\mathbf{u}\|^2$

➢ **Goal:** design a *computationally efficient* robust controller under *distributional uncertainty* of exogenous disturbances

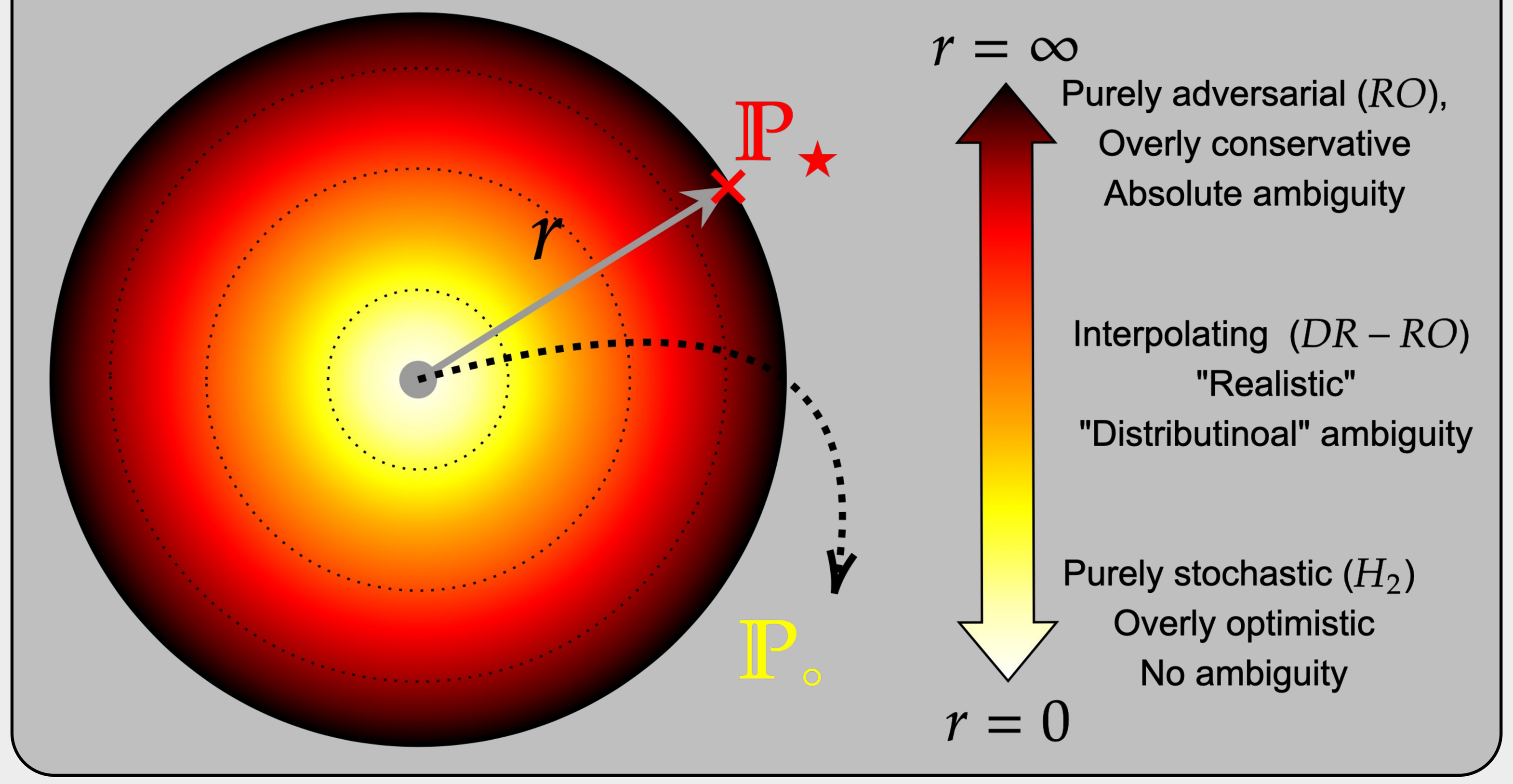
➢ **Performance metric:** regret against non-causal controller

$$\textsc{Regret}(\mathbf{u},\mathbf{w}) := \textsc{Cost}(\mathbf{u},\mathbf{w}) - \min_{\mathbf{u}_{nc}} \textsc{Cost}(\mathbf{u}_{nc},\mathbf{w})$$

$$\mathbf{u}_{nc} = \underbrace{-(\mathcal{I}+\mathcal{F}^\dagger\mathcal{F})^{-1}\mathcal{F}^\dagger\mathcal{G}}_{\text{"best" non-causal controller } \mathcal{K}_{nc}} \mathbf{w}$$

➢ **Disturbances:**

- *Unknown* disturbance dist.: $\mathbf{w} \sim \mathbb{P},\ \mathrm{Cov}[\mathbf{w}] = \mathcal{M}$
- Known *nominal dist.:* $\mathbf{w}_\circ \sim \mathbb{P}_\circ,\ \mathrm{Cov}[\mathbf{w}_\circ] = \mathcal{M}_\circ$
- Known Wasserstein-2 distance: $\mathrm{W}_2(\mathbb{P},\mathbb{P}_\circ) \le r$



➢ **Prior works:** Taskesen et al. 2023 restrict to time-independent disturbances, Taha et al. 2023 have similar setup but only for the finite-horizon setting

➢ **Challenge:** *Finite-horizon DR-RO controller requires solving an SDP scaling with time-horizon*. Therefore, we seek an infinite-horizon DR-RO controller.

➢ **Approach:** Take the limit of finite-horizon problem and formulate as *convex-concave optimization over Toeplitz, positive-definite, autocovariance operators.*

2. Main Results

➢ **Primal (P) problem:**

$$(\mathrm{P}) \quad \inf_{\text{causal } \mathcal{K}} \ \sup_{\mathbb{P}:\, \mathrm{W}_2(\mathbb{P},\mathbb{P}_\circ)\le r} \mathbb{E}_{\mathbf{w}\sim\mathbb{P}}\left[\textsc{Regret}(\mathcal{K}\mathbf{w},\mathbf{w})\right]$$

➢ (P) can be simplified since regret admits a quadratic form:

$$\textsc{Regret}(\mathcal{K}\mathbf{w},\mathbf{w}) = \mathbf{w}^\dagger \underbrace{(\Delta\mathcal{K}-\Delta\mathcal{K}_{nc})^\dagger(\Delta\mathcal{K}-\Delta\mathcal{K}_{nc})}_{\text{regret operator } \mathcal{R}_\mathcal{K}} \mathbf{w}$$

where $\Delta^\dagger\Delta = \mathcal{I} + \mathcal{F}^\dagger\mathcal{F}$ is the *spectral factorization.*

Theorem 1. *Problem (P) is equivalent to problem (D) below*

$$(\mathrm{D}) \quad \sup_{\substack{\mathcal{M}\succ 0,\\ \mathrm{BW}(\mathcal{M},\mathcal{M}_\circ)\le r}} \ \inf_{\text{causal } \mathcal{K}} \mathrm{Tr}(\mathcal{R}_\mathcal{K}\,\mathcal{M})$$

The saddle point $(\mathcal{K}_\star, \mathcal{M}_\star)$ *satisfies:*

i) $$\mathcal{K}_\star = \underbrace{\mathcal{K}_\circ}_{\text{Nominal Policy}} + \underbrace{\Delta^{-1}\{\{\Delta\mathcal{K}_{nc}\mathcal{L}_\circ\}_- \mathcal{L}_\circ^{-1}\mathcal{L}_\star\}_+ \mathcal{L}_\star^{-1}}_{\text{Additive Corrective Term}},$$

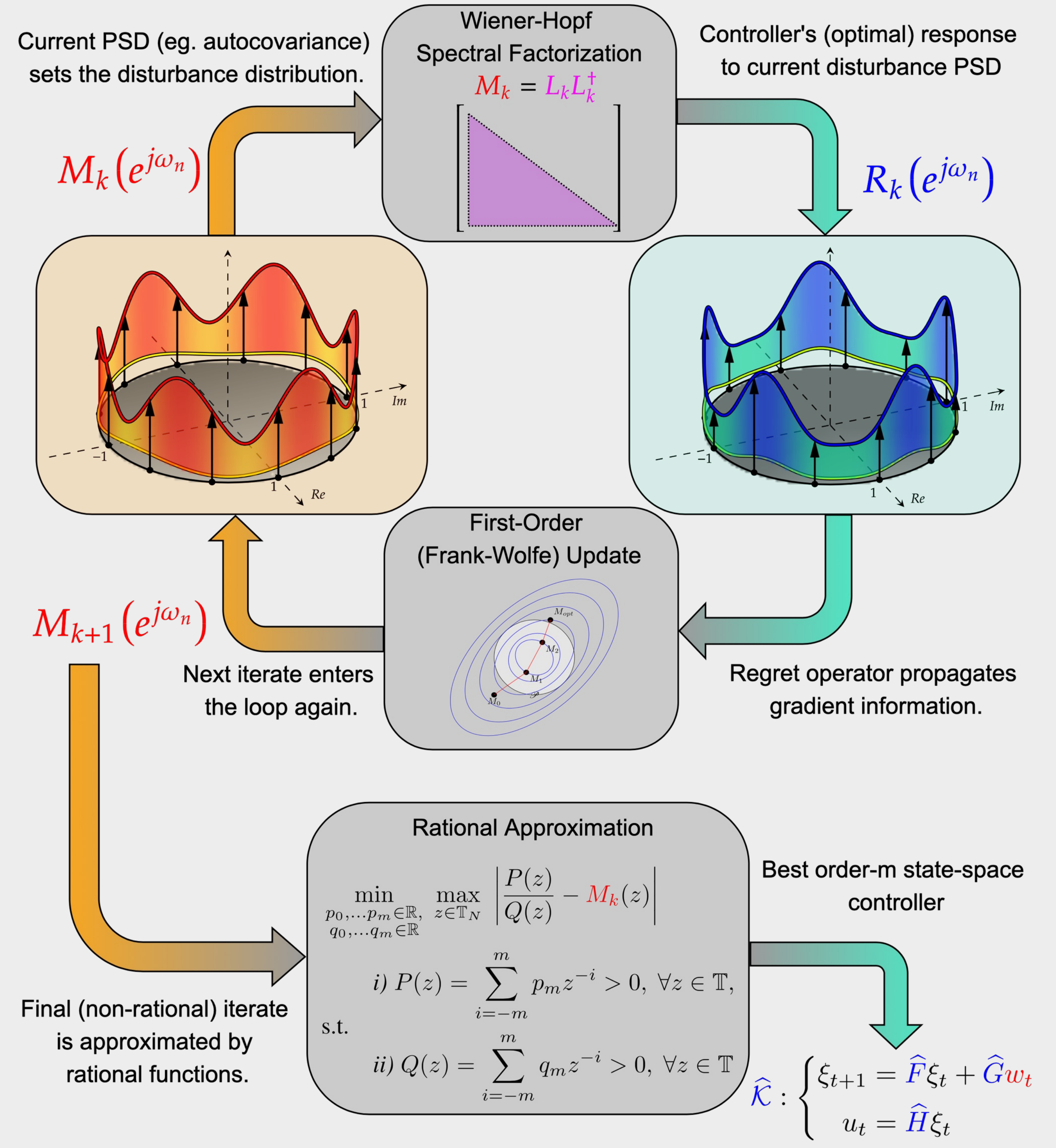
ii) $$\underbrace{\mathcal{M}_\star}_{\text{Worst-case Autocovariance}} = \underbrace{(\mathcal{I}-\gamma_\star^{-1}\mathcal{R}_{\mathcal{K}_\star})^{-1}}_{\text{Optimal Transport Map}} \underbrace{\mathcal{M}_\circ}_{\text{Nominal Autocovariance}} (\mathcal{I}-\gamma_\star^{-1}\mathcal{R}_{\mathcal{K}_\star})^{-1}$$

where $\mathcal{M}_\star = \mathcal{L}_\star\mathcal{L}_\star^\dagger$ *and* $\mathcal{M}_\circ = \mathcal{L}_\circ\mathcal{L}_\circ^\dagger$ *are Wiener-Hopf spectral factorizations and* $\gamma_\star > 0$ *is such that* $\mathrm{BW}(\mathcal{M}_\star,\mathcal{M}_\circ) = r$.

➢ **Worst-case disturbance:** $\mathbf{w}_\star = (\mathcal{I}-\gamma_\star^{-1}\mathcal{R}_{\mathcal{K}_\star})^{-1}\mathbf{w}_\circ$

Theorem 2. *The worst-case covariance* $\mathcal{M}_\star$ *and the optimal DR-RO controller* $\mathcal{K}_\star$ *are* ***non-rational****. Thus, optimal DR-RO controller does* ***not*** *admit a finite-order state-space realization.*

3. Algorithm



4. Numerical Simulations