Tight Partial Identification of Causal Effects with Marginal Distribution of Unmeasured Confounders (*Spotlight*)

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Introduction

Partial identification (PI) presents a significant challenge in causal inference due to the incomplete measurement of confounders. Given that obtaining auxiliary variables of confounders is not always feasible and relies on untestable assumptions, researchers are encouraged to explore the internal information of latent confounders without external assistance. However, these prevailing PI results often lack precise mathematical measurement from observational data or assume that the information pertaining to confounders falls within extreme scenarios. In our paper, we reassess the significance of the marginal confounder distribution in PI. We refrain from imposing additional restrictions on the marginal confounder distribution, such as entropy or mutual information. Instead, we establish the closed-form tight PI for any possible $\mathbf{P}(U)$ in the discrete case. Furthermore, we establish the if and only if criteria for discerning whether the marginal confounder information leads to non-vanilla PI regions. This reveals a fundamental negative result wherein the marginal confounder information minimally contributes to PI as the confounder's cardinality increases. Experiments support our theoretical findings.

Motivation

With $\mathbb{P}(U)$ at hand, previous literature usually lacks precise mathematical measurements from observational data or assumes that the information pertaining to confounders falls within extreme scenarios.

We provide a **"white-box" mapping** from the marginal distribution of confounders to casual queries. We provide an in-depth relationship between the subset-sum problem in TCS and partial identification in causal inference/economics.

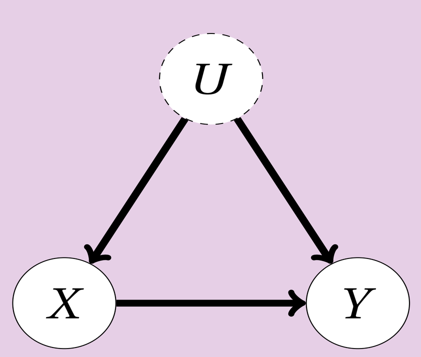

Figure 1. The causal graph constructed by treatment X, outcome Y, and confounders U. We focus on the tight partial identification of causal queries solely through the information of marginal distribution $\mathbb{P}(U)$ with observed $\mathbb{P}(X,Y)$.

Definition and Problem Formulation

When the information $\mathbb{P}(U)$ is not accessible, Judea Pearl shows that

$$\mathbb{P}(y \mid do(x)) = \mathbb{P}(x,y) + \sum_u \frac{\mathbb{P}(y,u,x)}{\mathbb{P}(u,x)}\mathbb{P}(u,\neg x) \qquad (1)$$

belongs to $[\mathbb{P}(x,y), \mathbb{P}(x,y)+\mathbb{P}(\neg x)]$, where since $x \in \{0,1\}$, we write $\neg x \equiv 1-x$, i.e., that $\mathbb{P}(\neg x) \equiv \mathbb{P}(X = 1-x)$. We denote $\mathbb{P}(x,y)$ and $\mathbb{P}(x,y)+\mathbb{P}(\neg x)$ as the "vanilla lower bound" and "vanilla upper bound" of $\mathbb{P}(y \mid do(x))$, respectively.

Stepping forward, we denote $-\mathbb{P}(X=1,Y=0)-\mathbb{P}(X=0,Y=1)$, $\mathbb{P}(X=1,Y=1)+\mathbb{P}(X=0,Y=0)$ and $\mathbb{P}(Y=1 \mid X=1)-\mathbb{P}(Y=1 \mid X=0)$ as the 'vanilla lower bound of ATE', 'vanilla upper bound of ATE''vanilla extreme bound of ATE'. Such lower and upper vanilla bounds have already been derived from previous literature.

Assumption

Positivity $\forall x \in \{0,1\}, y \in \{0,1\}, \mathbb{P}(x,y) > 0$. We also invoke the following assumption on the properties of $\mathbb{P}(U)$. Moreover, U is a discrete random variable taking values in $\{0,\dots,d_u-1\}$. Moreover, there does not exist a u' such that $\mathbb{P}(U=u')=1$.

Our setting could be easily generalized to the continuous cases, observed confounder cases, etc.

Theorem (Binary case & General case)

Theorem (Identification of interventional probability)

Suppose $d_u = 2$ and the distribution $\mathbb{P}(U)$ observable. The tight identification region of the interventional probability $\mathbb{P}(y \mid do(x))$ is given by

$$\left[\min_{t\in\{0,1\}} \mathcal{LB}\left(\mathbb{P}(U=t)\right), \max_{t\in\{0,1\}} \mathcal{UB}\left(\mathbb{P}(U=t)\right)\right].$$

Here $\mathcal{LB}(\cdot)$,$\mathcal{UB}(\cdot)$ are two piece-wise linear functions defined as

$$\begin{cases} \frac{\mathbb{P}(x,y)-t}{\mathbb{P}(x)-t}(1-t)+t & t \in (0,\mathbb{P}(x,y)] \\ \mathbb{P}(x,y) & t \in (\mathbb{P}(x,y),\mathbb{P}(x)] \\ \mathbb{P}(y \mid x)t & t \in (\mathbb{P}(x),1) \end{cases} \text{ and } \begin{cases} \mathbb{P}(y \mid x)(1-t)+t & t \in (0,\mathbb{P}(\neg x)] \\ \mathbb{P}(x,y)+\mathbb{P}(\neg x) & t \in (\mathbb{P}(\neg x),1-\mathbb{P}(x,\neg y)] \\ \frac{\mathbb{P}(x,y)t}{\mathbb{P}(x)-(1-t)} & t \in (1-\mathbb{P}(x,\neg y),1) \end{cases}. \qquad (2)$$

Theorem (Identification of average treatment effect)

Consider the same setup as Theorem 1, then the tight identification region of ATE is given by

$$\left[\min_{t\in\{0,1\}} \{-\mathcal{B}(\mathbb{P}(U=t);0,1)\}, \max_{t\in\{0,1\}} \mathcal{B}(\mathbb{P}(U=t);1,1)\right],$$

where $\mathcal{B}(t;x,y) :=$

$$\begin{cases} \left(-\mathbb{P}(y \mid \neg x)+\frac{\mathbb{P}(x,y)}{\mathbb{P}(x)-t}\right)(1-t) & t \in (0,p_0] \\ -\mathbb{P}(y \mid \neg x)(1-t)-\mathbb{P}(x,\neg y)+1 & t \in (p_0,p_1] \\ -\mathbb{P}(\neg x,y)+\mathbb{P}(y \mid x)t+(1-t) & t \in (p_1,p_2] \\ \left(-\frac{\mathbb{P}(\neg x,y)-(1-t)}{\mathbb{P}(\neg x)-(1-t)}+\mathbb{P}(y \mid x)\right)t & t \in (p_2,1) \end{cases}.$$

Here $p_0 = \mathbb{P}(x,\neg y)$, $p_1 = \mathbb{P}(x)$, $p_2 = 1-\mathbb{P}(\neg x,y)$.

Theorem (IFF condition for interventional probability)

The tight lower bound of the interventional probability $\mathbb{P}(y \mid do(x))$ given prior knowledge of $\mathbb{P}(U)$ is equal to the vanilla lower bound if and only if $\mathbb{P}(U)$ belongs to $\mathcal{P}^L :=$

$$\left\{\mathbb{P}(U) : \exists\, \mathcal{U} \subseteq \mathbb{R} \ s.t.\ \mathbb{P}(U \in \mathcal{U}) \in [\mathbb{P}(x,y),\mathbb{P}(x)]\right\}.$$

Analogously, the if and only if condition for the degeneration of upper bound is when $\mathbb{P}(U)$ belongs to $\mathcal{P}^U :=$

$$\left\{\mathbb{P}(U) : \exists\, \mathcal{U} \subseteq \mathbb{R} \ s.t.\ \mathbb{P}(U \in \mathcal{U}) \in [\mathbb{P}(\neg x),1-\mathbb{P}(x,\neg y)]\right\}.$$

Thus the tight identification region of $\mathbb{P}(y \mid do(x))$ given prior knowledge of $\mathbb{P}(U)$ is equal to the vanilla bound if and only if $\mathbb{P}(U) \in \mathcal{P} := \mathcal{P}^L \cap \mathcal{P}^U$.

Theorem (IFF condition for ATE)

The if and only if conditions for the tight upper and lower bounds of the average treatment effect to degenerate into vanilla bounds are when $\mathbb{P}(U)$ belongs to

$$\mathcal{P}^L_{\text{ATE}} := \{\mathbb{P}(U) : \exists\, \mathcal{U}_0,\mathcal{U}_1 \subseteq \mathbb{R} \textit{ with } \mathcal{U}_0 \cap \mathcal{U}_1 = \emptyset,\ s.t.\ \forall z \in \{0,1\}, \mathbb{P}(U \in \mathcal{U}_z) \in \mathcal{I}_{z,z}\},$$

where $\mathcal{I}_{x',y'} := [\mathbb{P}(X=x',Y=y'),\mathbb{P}(X=x')]$ for $x',y' \in \{0,1\}$ and

$$\mathcal{P}^U_{\text{ATE}} := \{\mathbb{P}(U) : \exists\, \mathcal{U}_0,\mathcal{U}_1 \subseteq \mathbb{R} \textit{ with } \mathcal{U}_0 \cap \mathcal{U}_1 = \emptyset,\ s.t.\ \forall z \in \{0,1\}, \mathbb{P}(U \in \mathcal{U}_z) \in \mathcal{I}_{\neg z,z}\},$$

respectively. Thus, the identification region of the average treatment effect is vanilla if and only if $\mathbb{P}(U) \in \mathcal{P}_{\text{ATE}} := \mathcal{P}^L_{\text{ATE}} \cap \mathcal{P}^U_{\text{ATE}}$.

Counter-Intuitive negative result

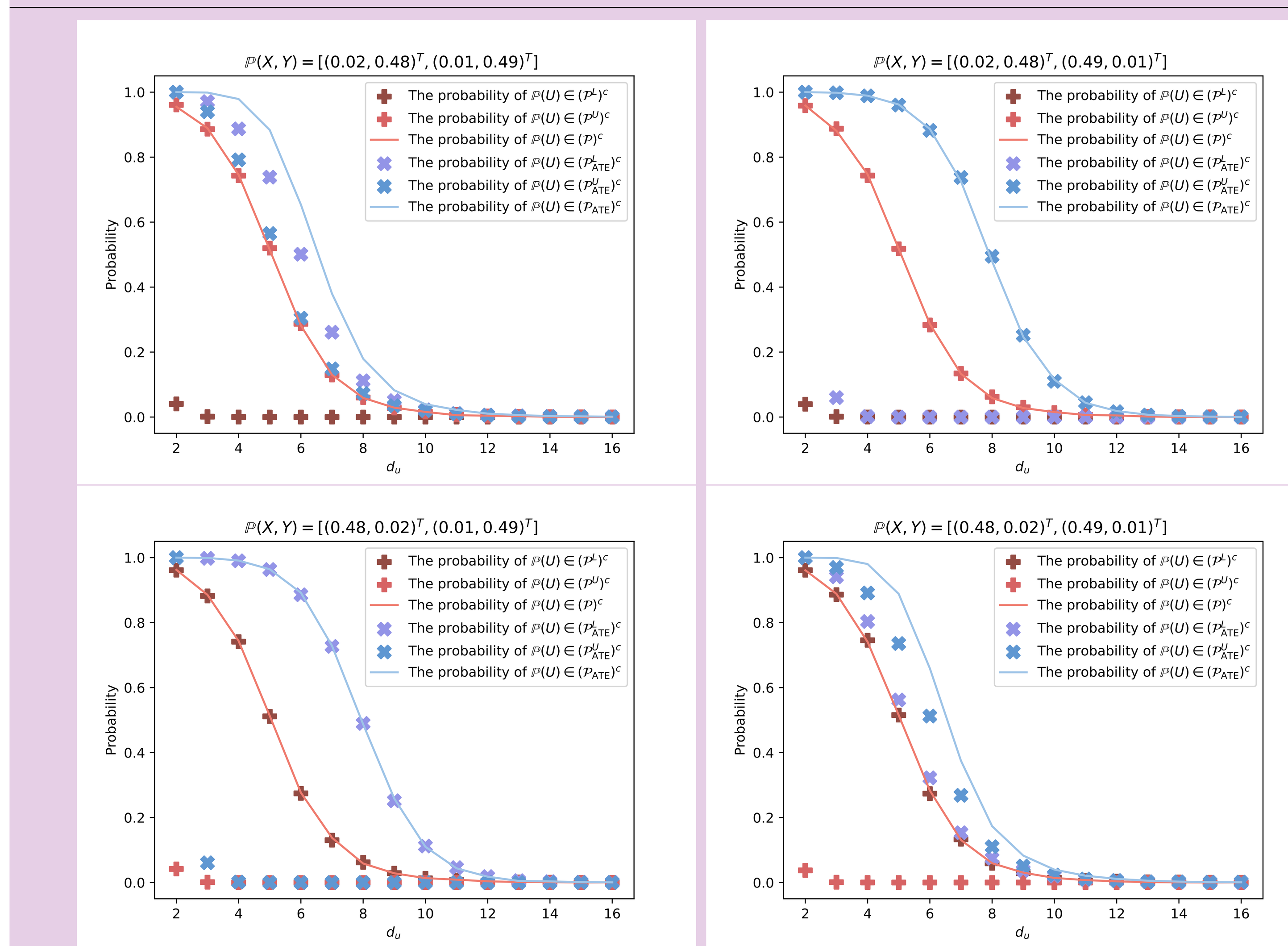

Figure 2. The probability that $\mathbb{P}(U)$ satisfies the if and only if condition given by Theorem 3 with varying d_u. Here $\mathbb{P}(U)$ is uniformly sampled on the $(d_u-1)-$ probability simplex via 10^6 Monte Carlo simulations. There are four types of observed data which are recorded as $\mathbb{P}(X,Y) = [(\mathbb{P}(X=1,Y=1),\mathbb{P}(X=1,Y=0))^T,(\mathbb{P}(X=0,Y=1),\mathbb{P}(X=0,Y=0))^T], x=y=1$. The probability of $\mathbb{P}(U) \in (\mathcal{P})^c$ and $\mathbb{P}(U) \in (\mathcal{P}_{\text{ATE}})^c$ both monotonically decreases to zero with increasing d_u, and the degeneration rate of $\mathbb{P}(U) \in (\mathcal{P})^c$ is lower than that of $\mathbb{P}(U) \in (\mathcal{P}_{\text{ATE}})^c$.

More results

Proposition

*The lower/upper tight identification bound of average treatment effect ATE is controlled by a **SSP** problem:*

$$\min \Big(|\mathbb{P}(U \in \mathcal{U}_0)-t_0| + |\mathbb{P}(U \in \mathcal{U}_1)-t_1|\Big), \quad s.t.\ \mathcal{U}_0,\mathcal{U}_1 \subseteq \mathbb{R}, \mathcal{U}_0 \cap \mathcal{U}_1 = \emptyset, t_0 \in \mathcal{I}, t_1 \in \mathcal{I}'. \qquad (3)$$