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GP PRIORS FOR SYSTEMS OF LINEAR PDE WITH CONSTANT COEFFICIENTS

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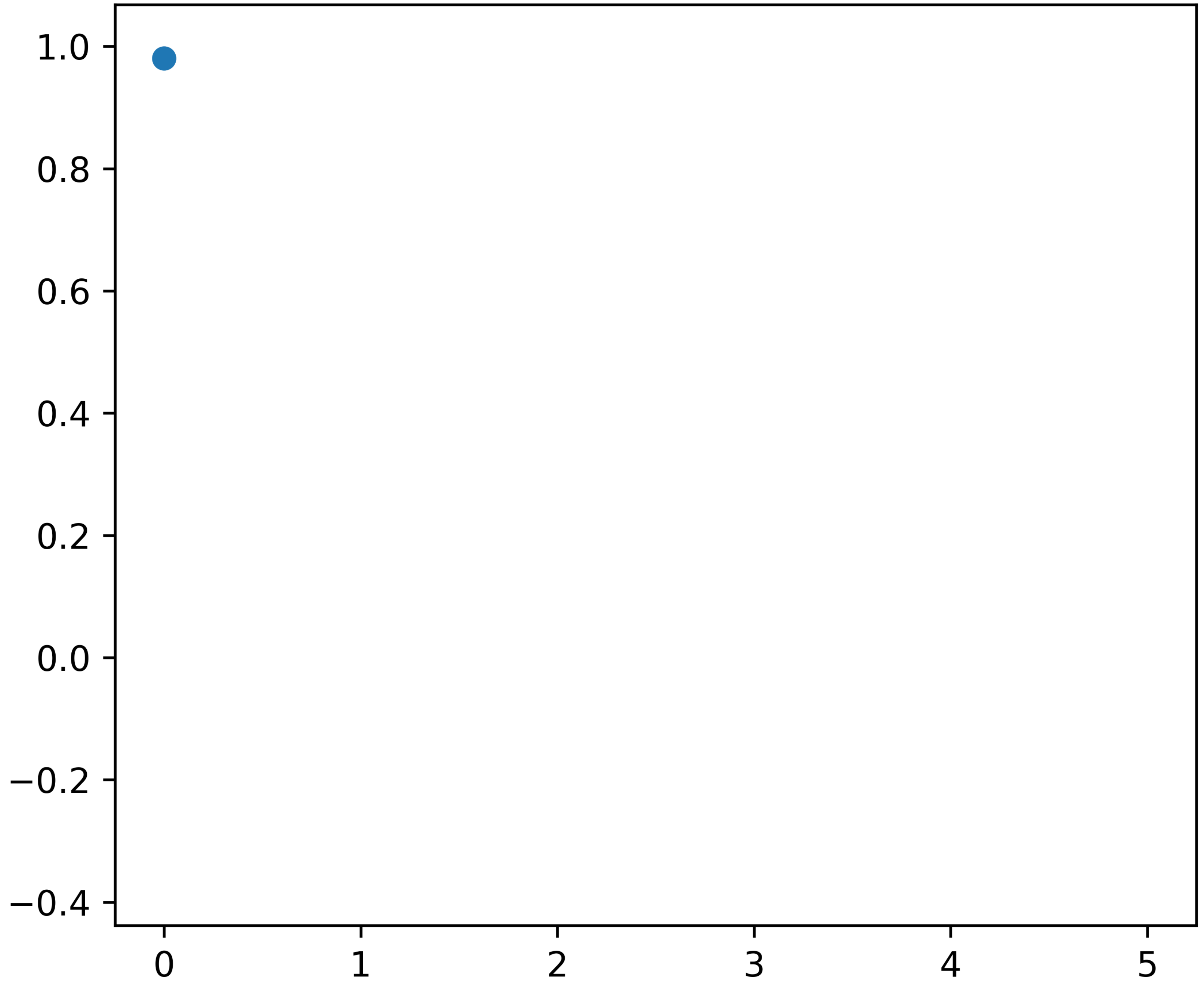
DAMPED SPRING-MASS SYSTEM

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 10y = 0$$

Sample 5 noisy points

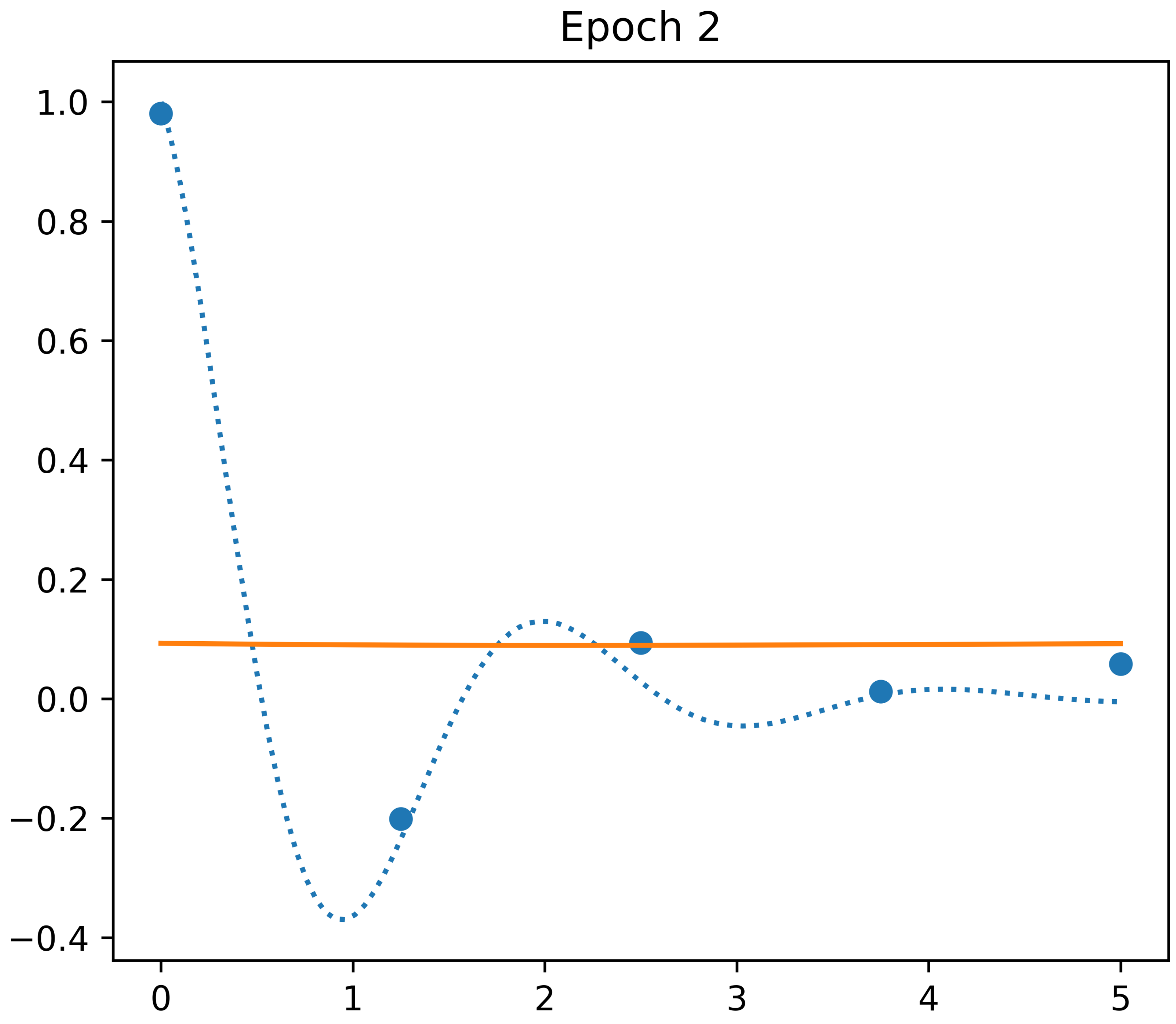


NEURAL NETWORK

⚠️~300,000 parameters

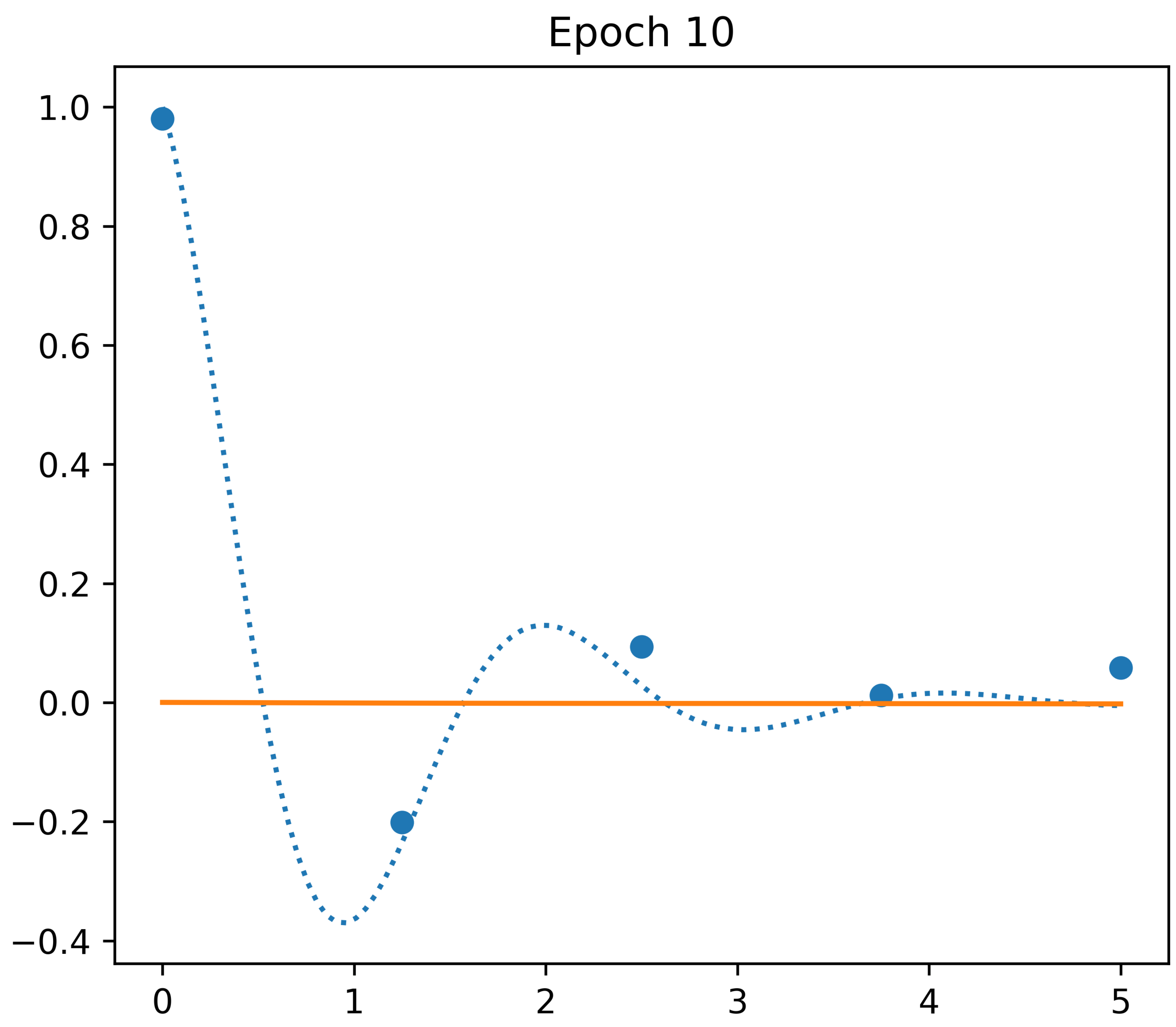
✅Fast convergence

❌Bad inter/extrapolation



PHYSICS INFORMED NEURAL NETWORK

- ⚠️~300,000 parameters
- ❌Slow convergence
- ✅Good inter/extrapolation
- ⚠️Requires extra points
- ⚠️Not an exact solution
- ❌Hard to optimize



How to solve?

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Algebraic preprocessing

$$z^2 + 2z + 10 = 0 \Rightarrow z = 1 \pm 3i$$

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Solution Space

$$y(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

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Gaussian process prior

$$Y(t) \sim e^{-t}(C_1 \cos 3t + C_2 \sin 3t) + \epsilon$$

where

$$C_1 \sim \mathcal{N}(0, \sigma_1), C_2 \sim \mathcal{N}(0, \sigma_2), \epsilon \sim \mathcal{N}(0, \sigma_0)$$

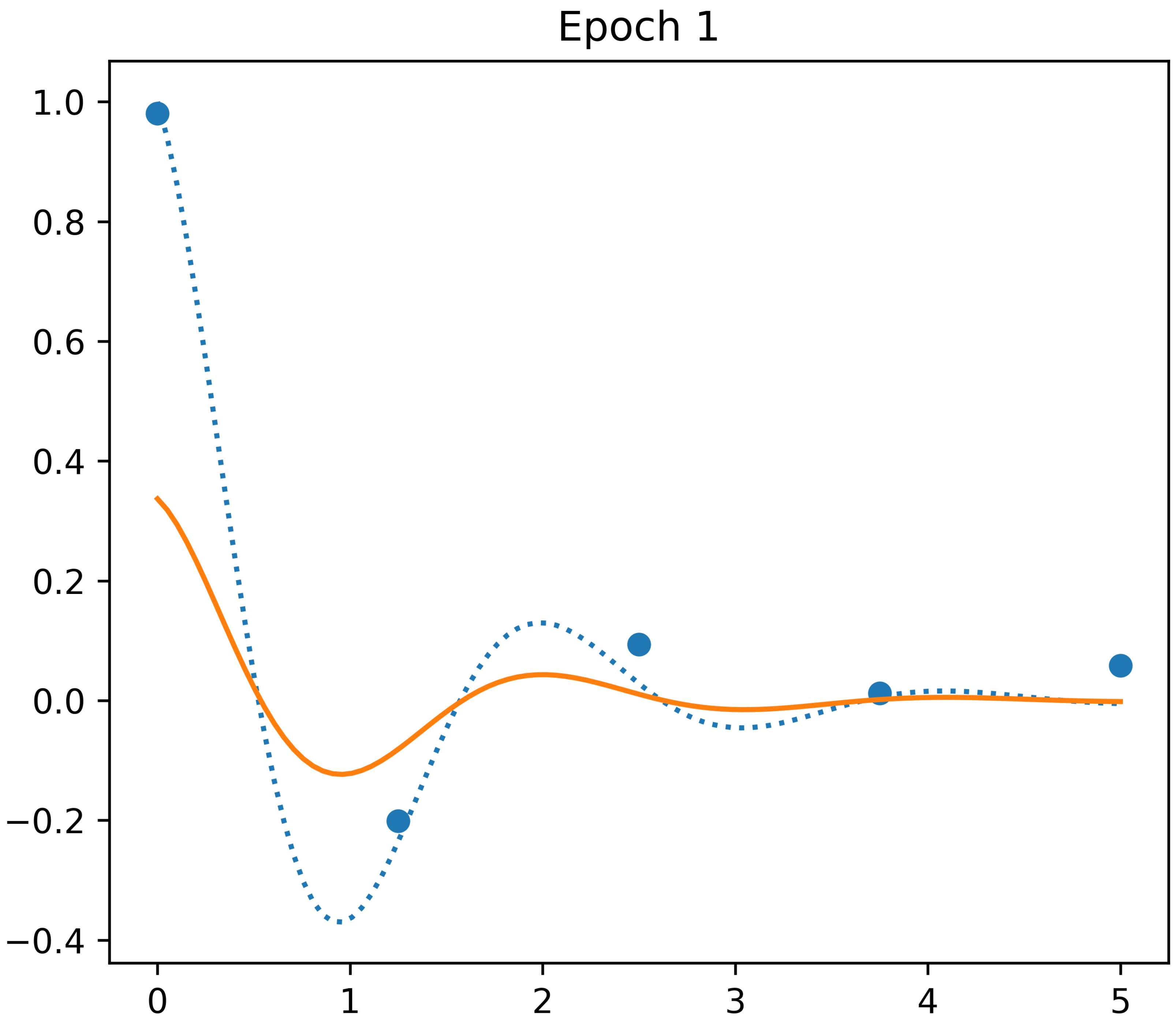
✅3 parameters

✅Fast convergence

✅Good inter/extrapolation

✅Exact solution

✅No extra points



Ehrenpreis-Palamodov ('70): Solutions to PDE described by "Fourier" frequencies

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Learned

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Works with **arbitrary** systems of homogeneous linear PDE with constant coefficients

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EPGP: Gaussian Process kernels *constrained* to PDE solutions

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

$\mathbf{E}(x, y, z, t)$ $\mathbf{B}(x, y, z, t)$

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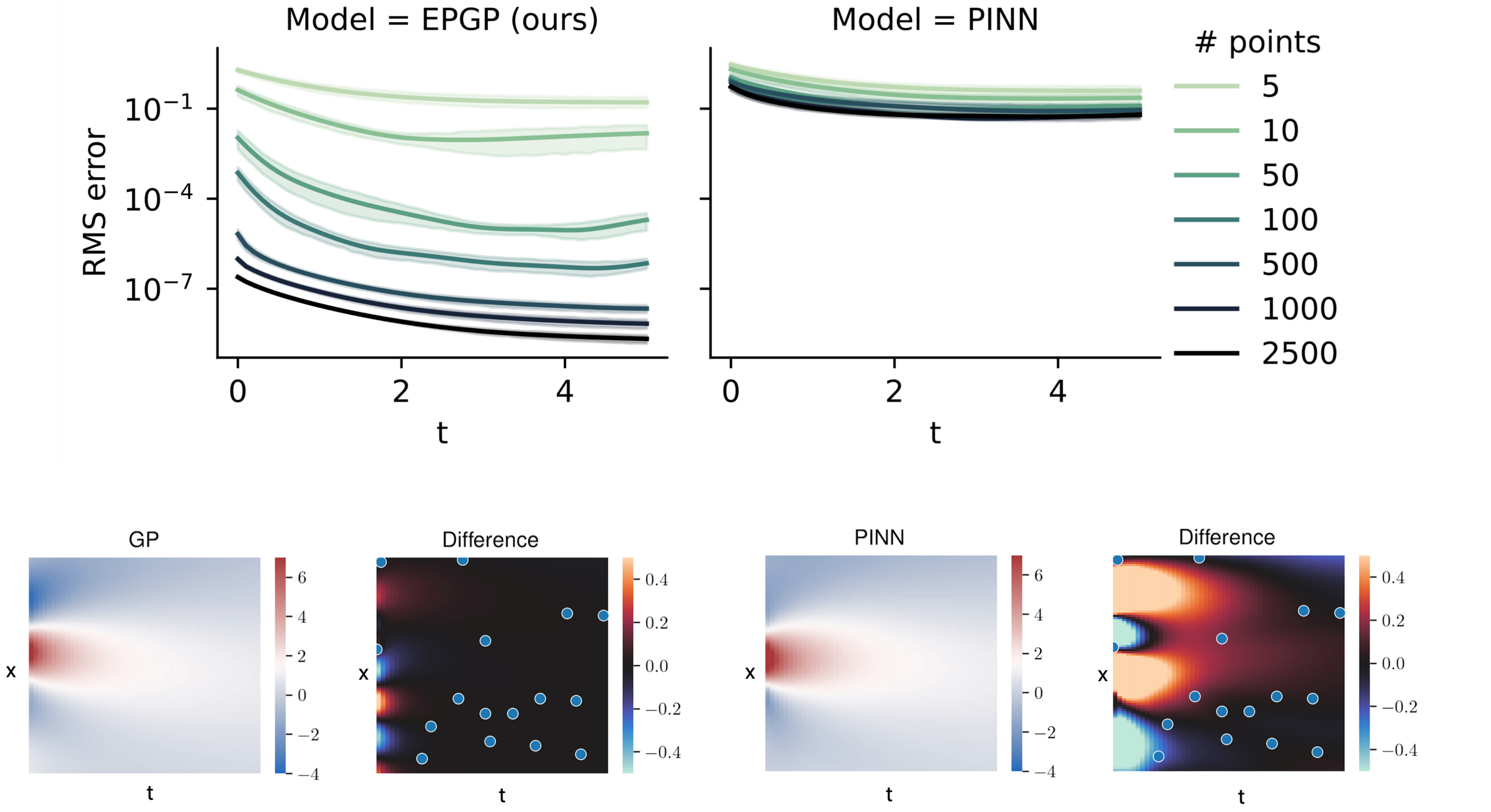
ROOT MEAN SQUARE ERRORS

	Datapoints					
Model	**5**	**10**	**50**	**100**	**1000**	**Training time (s)**
EPGP (Ours)	3.5	2.4	0.19	0.0097	0.00047	70
PINN	4.72	4.1	1.06	0.73	0.092	200


Model = EPGP (ours)
Model = PINN
RMS error
10^{-1}
10^{-4}
10^{-7}
0
2
4
t
points
5
10
50
100
500
1000
2500
GP
Difference
PINN
Difference
x
t

- **EPGP**: A class of Gaussian Process priors for solving PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

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  - **S-EPGP**: Sparse version

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Paper website