The Power of First-Order Smooth Optimization for Black-Box Non-Smooth Problems


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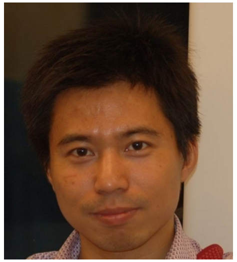
Bin Gu
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Optimization problem

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By solving a non-smooth problem, we run the first order SGD for a smooth problem that is very close to the original problem.

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In our case

$$L \leq \frac{2\sqrt{d}MM_2}{\varepsilon}$$

$$\sigma^2 \leq 2\kappa(p,d)dM_2^2$$

$$\kappa(p,d) = O\left(\min\{q, \ln d\}\, d^{2/q-1}\right)$$

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Gradient-free methods for non-smooth problems can be analysed in a unified way if first order smooth stochastic analysis is already available.