The Algebraic Path Problem for Graph Metrics

Enrique Fita Sanmartín

Sebastian Damrich

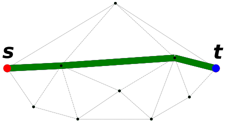
Fred A. Hamprecht

IWR at Heidelberg University

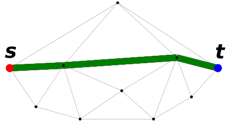
Interdisciplinary Center for Scientific Computing

UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

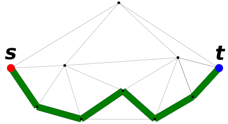
Shortest path distance



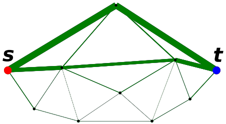
Shortest path distance



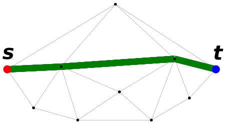
Minimax distance



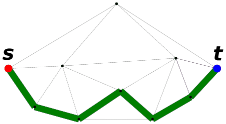
Commute cost distance



Shortest path distance

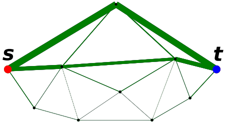


Minimax distance



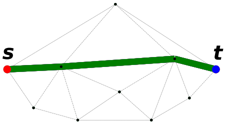
Eisner semiring

Commute cost distance



Min-plus semiring

Shortest path distance



Shortest Path Problem

$$\min_{\pi \in \mathcal{P}_{st}} \sum_{e \in \pi} c(e)$$

$\min \to \oplus$

$+ \to \otimes$

Algebraic Path Problem

$$\bigoplus_{\pi \in \mathcal{P}_{st}} \bigotimes_{e \in \pi} c(e)$$

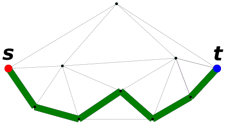
$\bigotimes \to$ edge concatenation operation

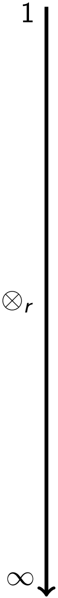
$\bigoplus \to$ path aggregator operation

} semiring

Minimax semiring

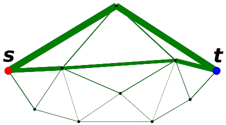
Minimax distance





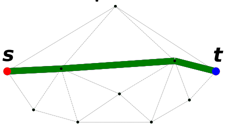
Eisner semiring

Commute cost distance



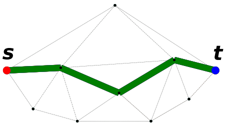
Min-plus semiring

Shortest path distance



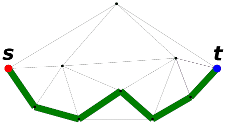
Min-norm semiring

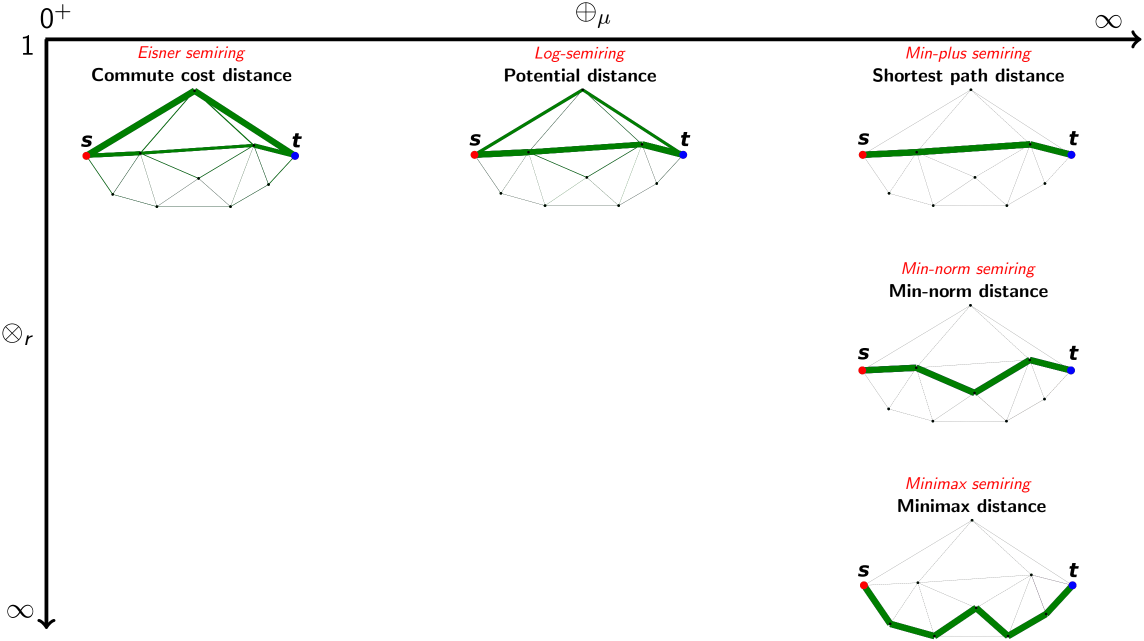
Min-norm distance



Minimax semiring

Minimax distance





0^+
$\bigoplus_\mu$
∞
1
Eisner semiring
Commute cost distance
Log-semiring
Potential distance
Min-plus semiring
Shortest path distance
Min-norm semiring
Min-norm distance
Minimax semiring
Minimax distance
$\bigotimes_r$
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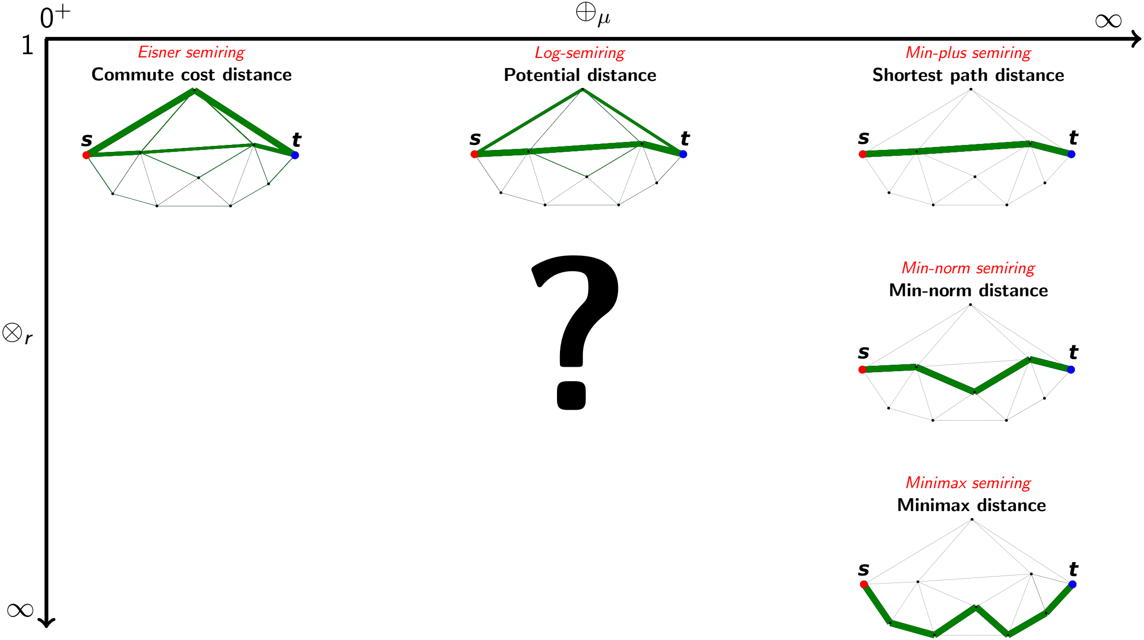
s
t



0^+
$\oplus_\mu$
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Eisner semiring
Commute cost distance
Log-semiring
Potential distance
Min-plus semiring
Shortest path distance
Min-norm semiring
Min-norm distance
Minimax semiring
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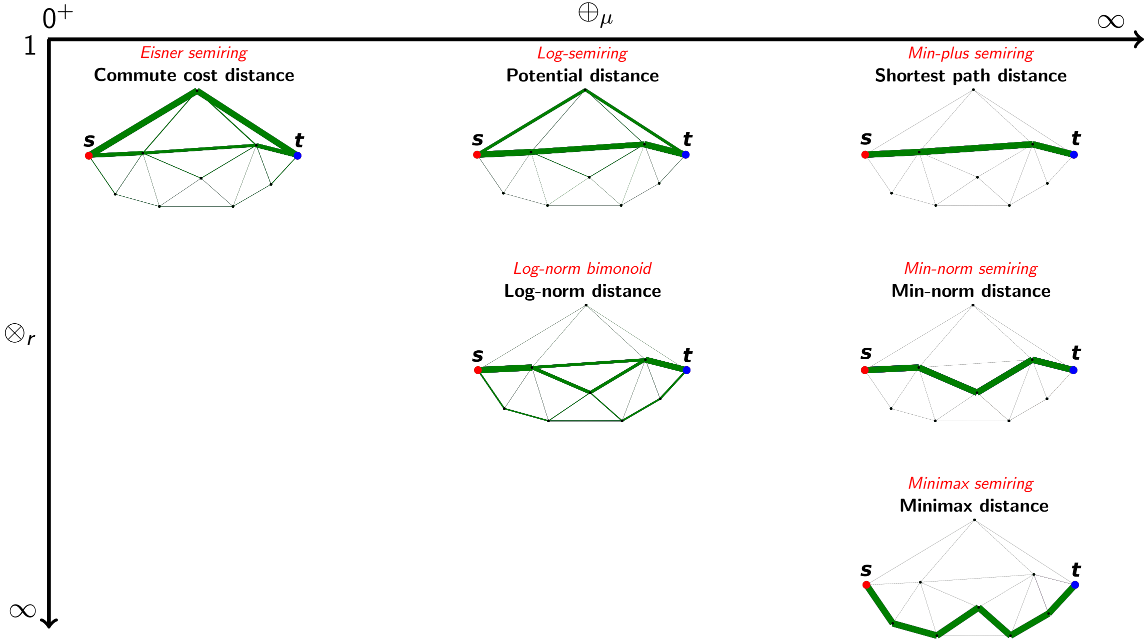

0^+
$\bigoplus_\mu$
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Eisner semiring
Commute cost distance
Log-semiring
Potential distance
Min-plus semiring
Shortest path distance
Log-norm bimonoid
Log-norm distance
Min-norm semiring
Min-norm distance
Minimax semiring
Minimax distance
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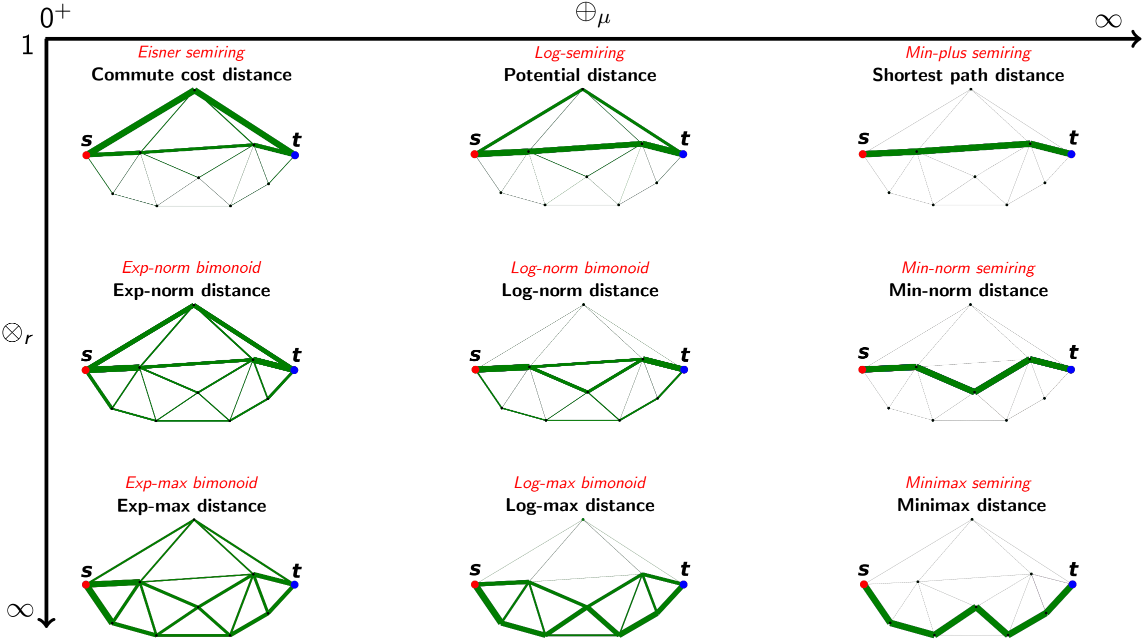
0^+
$\oplus_\mu$
∞
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$\otimes_r$
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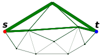
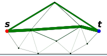
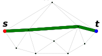
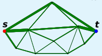
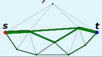
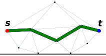
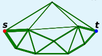
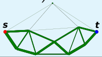
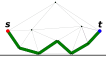
Eisner semiring
Commute cost distance
Log-semiring
Potential distance
Min-plus semiring
Shortest path distance
Exp-norm bimonoid
Exp-norm distance
Log-norm bimonoid
Log-norm distance
Min-norm semiring
Min-norm distance
Exp-max bimonoid
Exp-max distance
Log-max bimonoid
Log-max distance
Minimax semiring
Minimax distance
s
t

$r \backslash \mu$	0^+	$(0,\infty)$	∞
1	*Eisner semiring* **Commute cost distance** $\underset{\pi\sim\mathcal{P}^h_{st}}{\mathbb{E}}[c(\pi)] + \underset{\pi\sim\mathcal{P}^h_{ts}}{\mathbb{E}}[c(\pi)]$ Klein and Randić (1993) 	*Log-semiring* **Potential distance** $-\frac{1}{\mu}\left(\log\left(\underset{\pi\sim\mathcal{P}^h_{st}}{\mathbb{E}}\left[e^{-\mu c(\pi)}\right]\right) + \log\left(\underset{\pi\sim\mathcal{P}^h_{ts}}{\mathbb{E}}\left[e^{-\mu c(\pi)}\right]\right)\right)$ Kivimäki et al. (2014); Françoisse et al. (2017) 	*Min-plus semiring* **Shortest path distance** $\min_{\pi\in\mathcal{P}_{st}} c(\pi)$ 
$(1,\infty)$	*Exp-norm bimonoid* **Exp-norm distance** $\underset{\pi\sim\mathcal{P}^h_{st}}{\mathbb{E}}[\|\|c(\pi)\|\|_r] + \underset{\pi\sim\mathcal{P}^h_{ts}}{\mathbb{E}}[\|\|c(\pi)\|\|_r]$ 	*Log-norm bimonoid* **Log-norm distance** $-\frac{1}{\mu}\left(\log\left(\underset{\pi\sim\mathcal{P}^h_{st}}{\mathbb{E}}\left[e^{-\mu\|\|c(\pi)\|\|_r}\right]\right) + \log\left(\underset{\pi\sim\mathcal{P}^h_{ts}}{\mathbb{E}}\left[e^{-\mu\|\|c(\pi)\|\|_r}\right]\right)\right)$ 	*Min-norm semiring* **Min-norm distance** $\min_{\pi\in\mathcal{P}_{st}} \|\|c(\pi)\|\|_r$ Mckenzie and Damelin (2019) 
∞	*Exp-max bimonoid* **Exp-max distance** $\underset{\pi\sim\mathcal{P}^h_{st}}{\mathbb{E}}\left[\max_{e\in\pi} c(e)\right] + \underset{\pi\sim\mathcal{P}^h_{ts}}{\mathbb{E}}\left[\max_{e\in\pi} c(e)\right]$ 	*Log-max bimonoid* **Log-max distance** $-\frac{1}{\mu}\left(\log\left(\underset{\pi\sim\mathcal{P}^h_{st}}{\mathbb{E}}\left[e^{-\mu \max_{e\in\pi} c(e)}\right]\right) + \log\left(\underset{\pi\sim\mathcal{P}^h_{ts}}{\mathbb{E}}\left[e^{-\mu \max_{e\in\pi} c(e)}\right]\right)\right)$ 	*Minimax semiring* **Minimax distance** $\min_{\pi\in\mathcal{P}_{st}} \max_{e\in\pi} c(e)$ Maggs and Plotkin (1988) 

$$||c(\pi)||_r = \sqrt[r]{\textstyle\sum_{e\in\pi} c(e)^r}$$

References

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