# Adaptive Conformal Predictions for Time Series

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39th International Conference on Machine Learning




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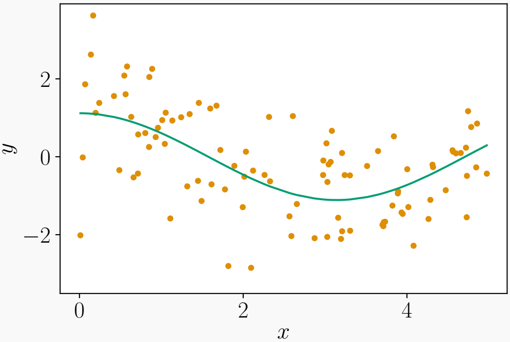
# Usual statistical learning



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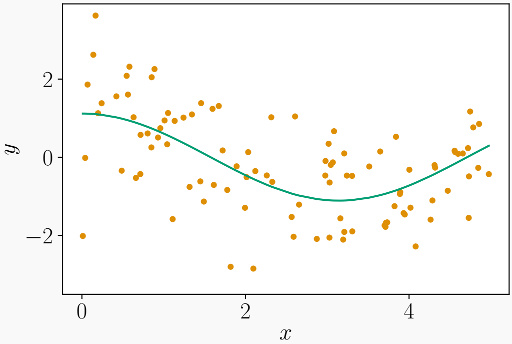
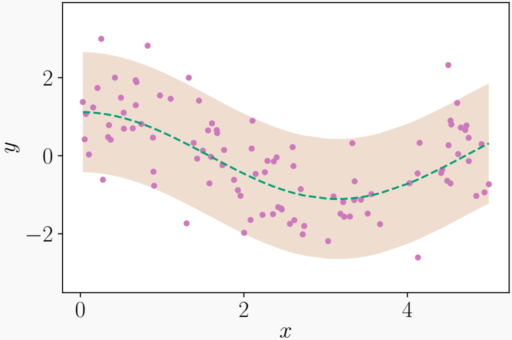



Unquantified uncertainty ⇒ incapacity of knowing if you can trust these predictions

# Split conformal prediction



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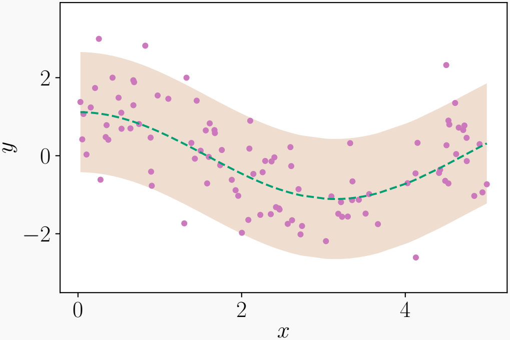


$$\mathbb{P}\left\{Y_{n+1} \in \text{Interval}_{\alpha}\left(X_{n+1}\right)\right\} \geq 1-\alpha$$

For example: $\alpha = 0.1$ and obtain a 90% coverage interval.

# Conformal prediction: summary

$$\mathbb{P}\left\{Y_{n+1} \in \mathrm{Interval}_{\alpha}\left(X_{n+1}\right)\right\} \geq 1 - \alpha$$

Split conformal prediction is simple to compute and works:

- finite sample;
- any regression algorithm (neural nets, random forest...);
- distribution-free as long as the data is exchangeable.

# Time series are not exchangeable

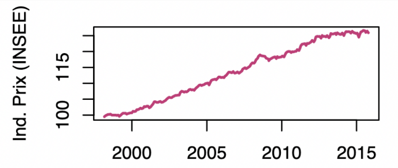


**Figure 1:** Trend[1]

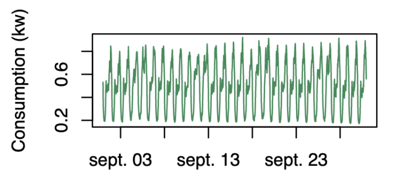


**Figure 2:** Seasonality[2]

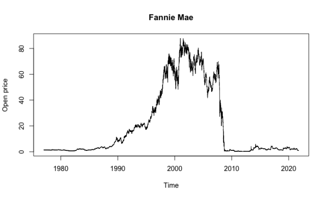


**Figure 3:** Shift

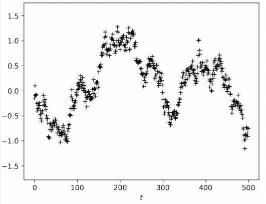


**Figure 4:** Time dependence

[1]Images from Yannig Goude class material.

# Framework and notations

- Data: $T_0$ observations $(x_1, y_1), \ldots, (x_{T_0}, y_{T_0})$ in $\mathbb{R}^d \times \mathbb{R}$
- Aim: predict the response values as well as predictive intervals for $T_1$ subsequent observations $x_{T_0+1}, \ldots, x_{T_0+T_1}$

$\hookrightarrow$ Build the smallest interval $\text{Interval}_\alpha^t$ such that:

$$\mathbb{P}\left\{Y_t \in \text{Interval}_\alpha^t\left(X_t\right)\right\} \geq 1-\alpha, \quad \text{for } t \in [\![T_0+1, T_0+T_1]\!].$$

# Extensions of conformal prediction to forecasting time series

- Chernozhukov et al. (2018)
- Wisniewski et al. (2020) and Kath and Ziel (2021)
- Xu and Xie (2021)
- Gibbs and Candès (2021)

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The proposed update scheme is the following:

$$\alpha_{t+1} := \alpha_t + \gamma\left(\alpha - \text{error}_t\right)$$

with some chosen $\gamma \geq 0$.

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Gibbs and Candès (2021) provide an asymptotic validity result for any distribution.

$$\frac{1}{T_1} \sum_{t=T_0+1}^{T_0+T_1} \mathbb{1} \left\{ y_t \in \text{Interval}_t(x_t) \right\} \underset{T_1 \to +\infty}{\longrightarrow} 1 - \alpha \quad \textit{e.g. 90\%}$$

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$\Rightarrow$ favors large $\gamma$. But, the higher $\gamma$, the more frequent are the infinite intervals.

# Theoretical analysis of ACI's length

# Approach

- Consider extreme cases (useful in an adversarial context) with simple theoretical distributions (additional assumptions);

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- Assume the calibration is perfect (and more), to rely on Markov Chain theory.

# Theoretical analysis of ACI's length: exchangeable case

**Theorem (Informal)**

*If the data is exchangeable and if the calibration is perfect, then as $\gamma \to 0$:*

$$\text{Average length of intervals from ACI using } \gamma$$
$$=$$
$$\text{Average length of intervals from Split Conformal Prediction}$$
$$+ \gamma \times \mathcal{C}(\alpha, \text{distribution of the data}),$$

*where $\mathcal{C}(\alpha, \text{distribution of the data}) > 0$ in non-atypical cases.*

# Theoretical and numerical analysis of ACI's length: AR(1) case

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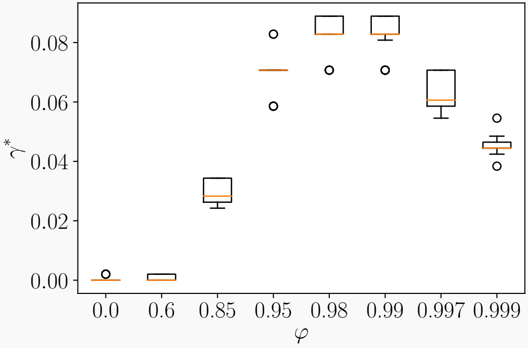


**Figure 5:** $\gamma^*$ minimizing the average length for each $\varphi$.

# AgACI

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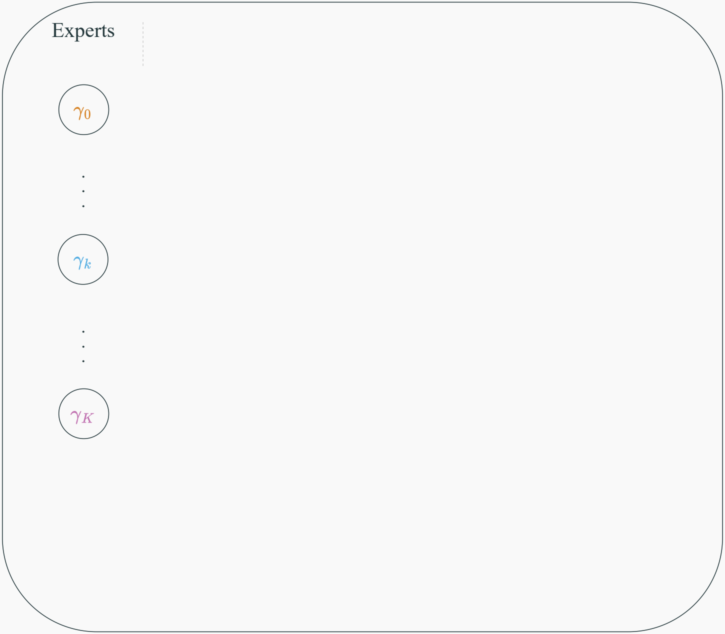
`AgACI` performs 2 independent aggregations: one for each bound (the upper and lower ones).

# AgACI: adaptive wrapper around ACI, scheme (upper bound)



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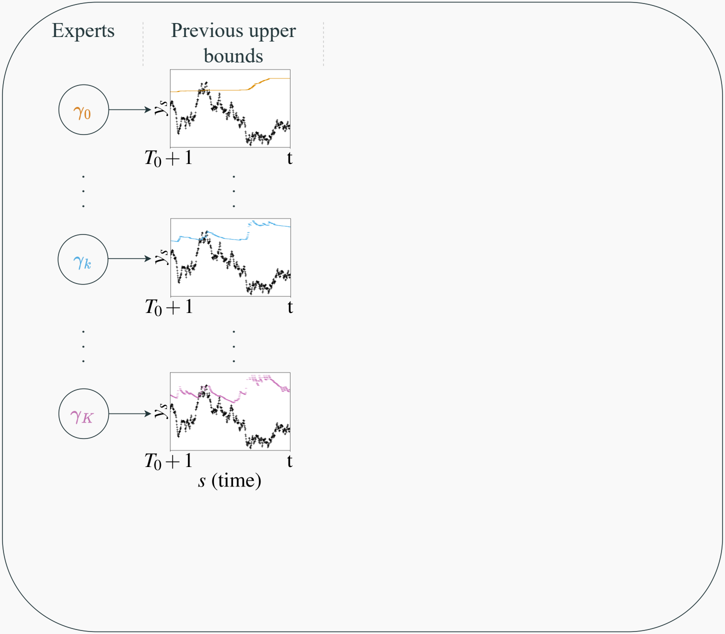

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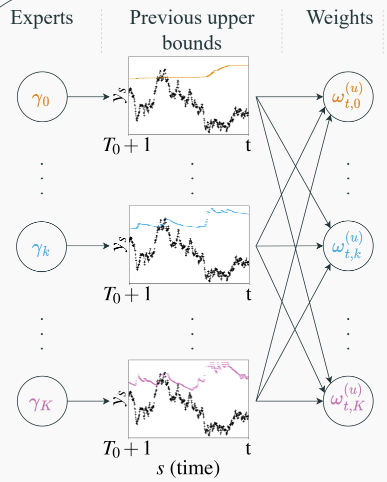

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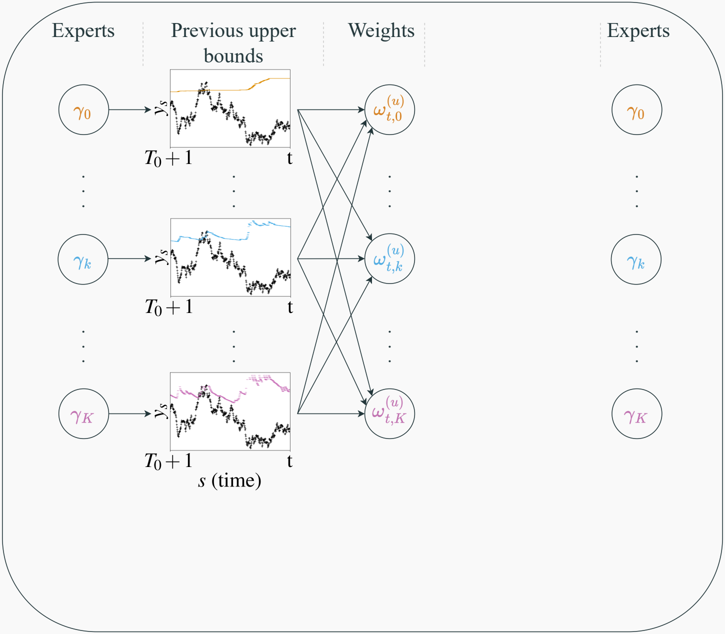

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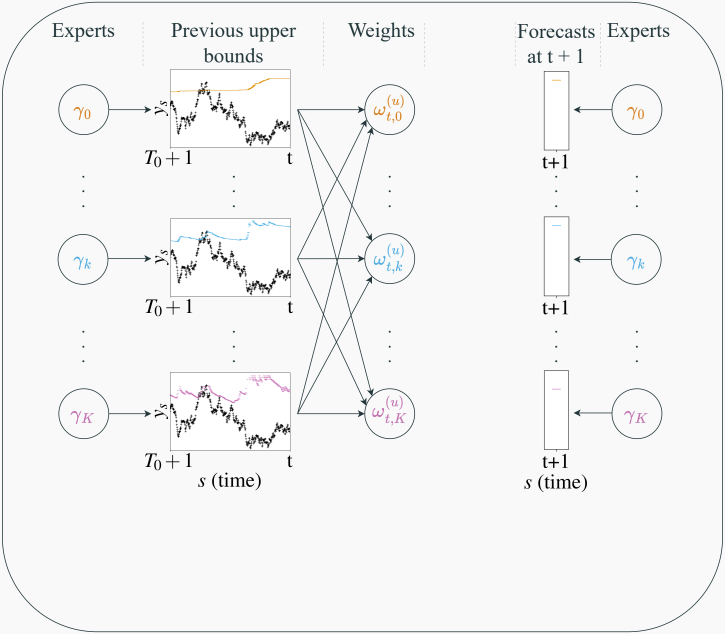

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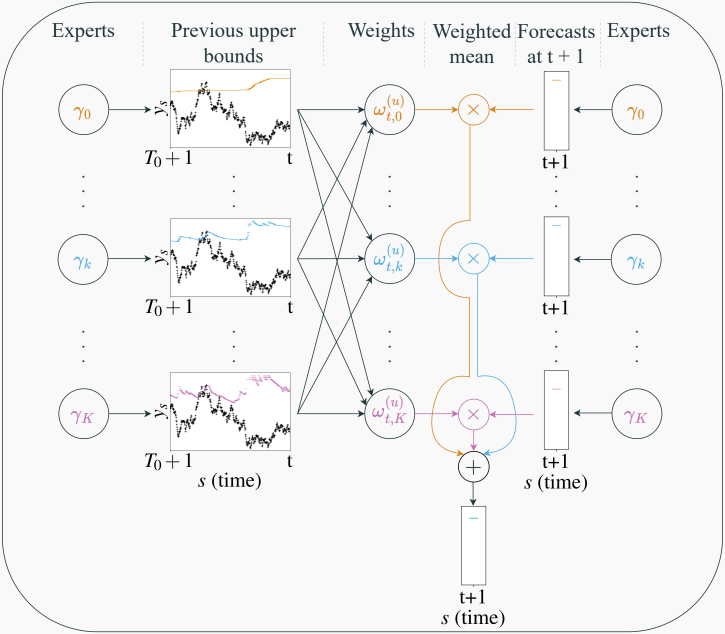

# Numerical experiments

## Simulated data and French electricity price forecasting

# Experimental take-away messages

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- Benchmarks are not robust to the increase in the temporal dependence;
- ACI is robust, maintaining validity, with an appropriate $\gamma$;
- AgACI is robust, maintaining validity, not the smallest;
- more in the paper!

**Thanks for listening!**

**To join us at the poster session: #117**