Bayesian Optimization for Distributionally Robust Chance-constrained Problem

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Black-box Function under Uncertainty Environment

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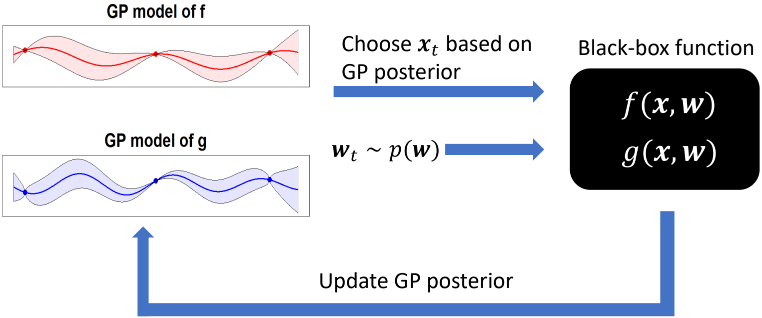

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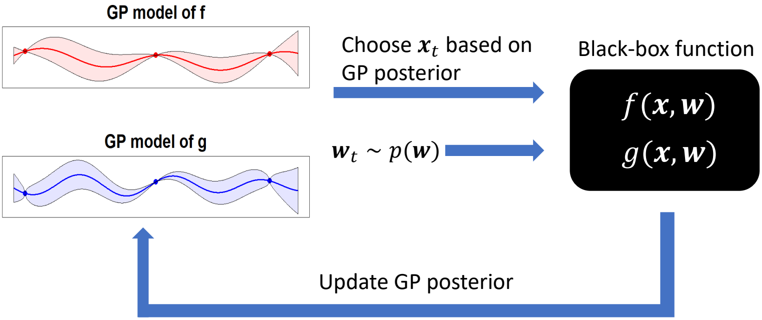

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  - **Environmental variable** w: Realized randomly.

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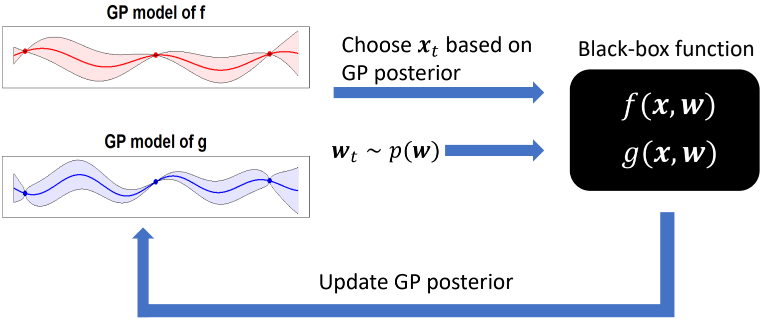

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  - **Design variable** x: Fully controllable.
  - **Environmental variable** w: Realized randomly.

Goal: Identify desired x by taking into account the uncertainty of w.

Robustness Measure

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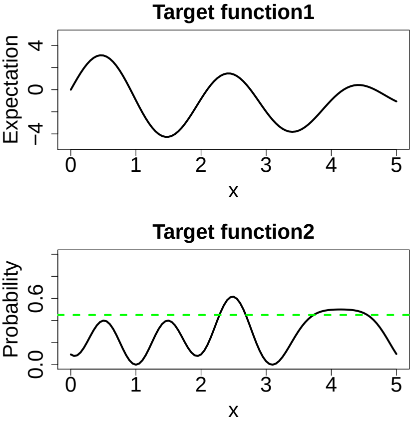
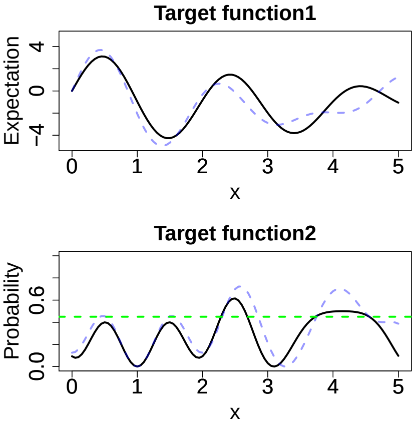

Target function1
Expectation
4
0
−4
0
1
2
3
4
5
x
Target function2
Probability
0.0
0.6
0
1
2
3
4
5
x

Problem Setup (DRCC Problem)



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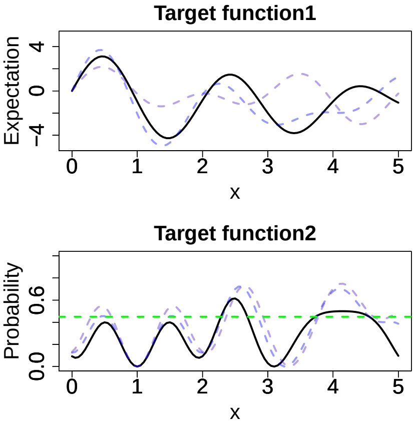

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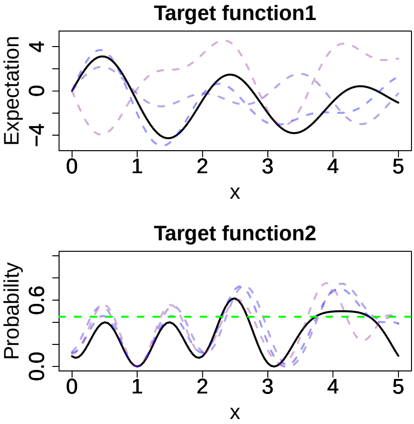

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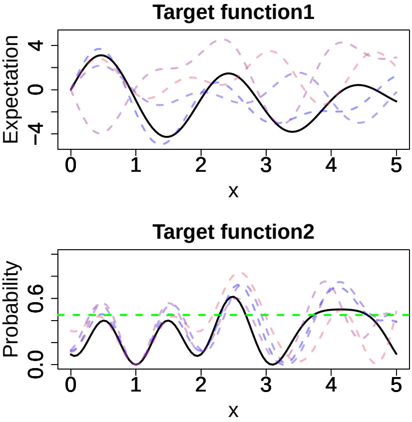

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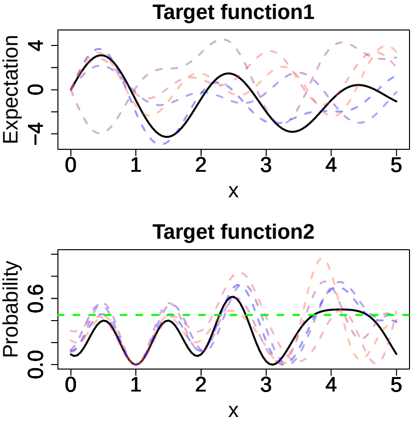

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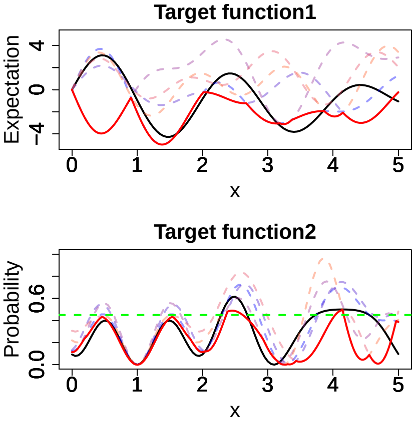

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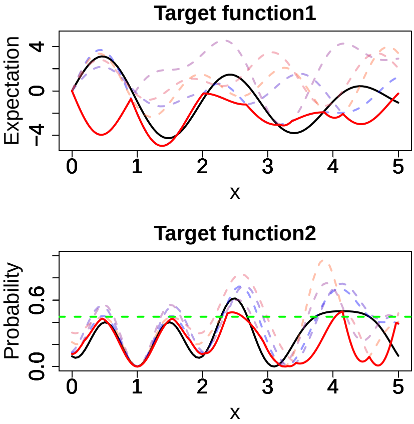

- **Target functions**:

$$F(\boldsymbol{x}) = \inf_{\mathbb{P}_{\boldsymbol{w}} \in \mathcal{A}} \mathbb{E}_{\boldsymbol{w} \sim \mathbb{P}_{\boldsymbol{w}}}[f(\boldsymbol{x}, \boldsymbol{w})],$$

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  - $\mathcal{A} \equiv \{\mathbb{P}_{\boldsymbol{w}} \mid d(\mathbb{P}_{\boldsymbol{w}}, \mathbb{P}_{\text{ref}}) < \epsilon\}$: Family of candidate distributions.
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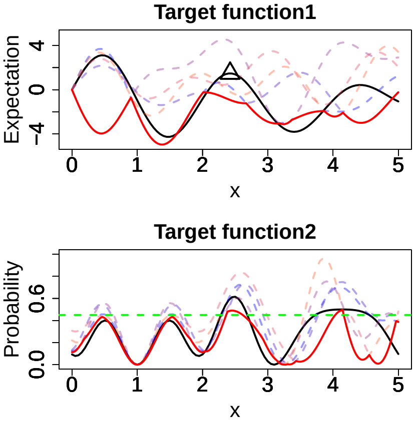

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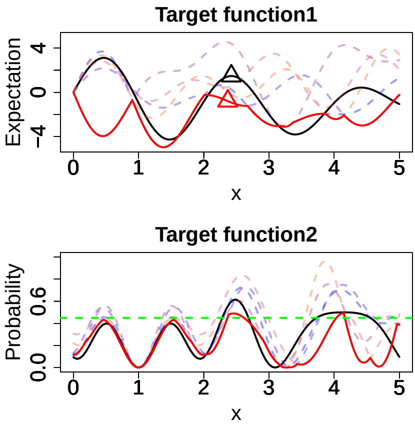
- $\triangle$ is the optimal value of CC problem.



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Goal: Find the optimal value of DRCC $\triangle$ efficiently.

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Basic idea of our proposed method

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  (Accuracy for DRCC): Estimated solution $\boldsymbol{x}_{\text{est,DRCC}}$ satisfies

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(Accuracy for CC): $\boldsymbol{x}_{\text{est,DRCC}}$ is a good solution of the CC problem:

$$\mathbb{E}_{\boldsymbol{w} \sim \mathbb{P}_{\boldsymbol{w}}^{(\text{true})}}[f(\boldsymbol{x}_{\text{true,CC}}, \boldsymbol{w})] - \mathbb{E}_{\boldsymbol{w} \sim \mathbb{P}_{\boldsymbol{w}}^{(\text{true})}}[f(\boldsymbol{x}_{\text{est,DRCC}}, \boldsymbol{w})] < 1.5\eta,$$
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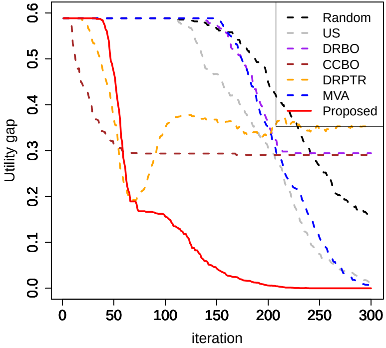
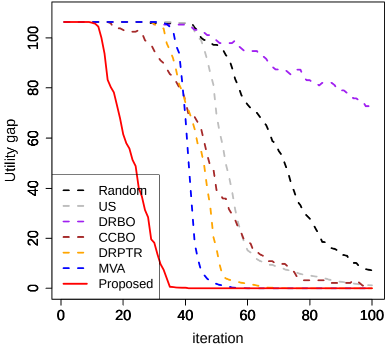
- Evaluate the performance of our proposed method synthetic and SIR simulation data.
- Measure the performance based on the utility gap.



Synthetic



SIR simulation

- **Solid lines** represent our proposed methods.