Imperial College London

Learning to Estimate and Refine Fluid Motion with Physical Dynamics

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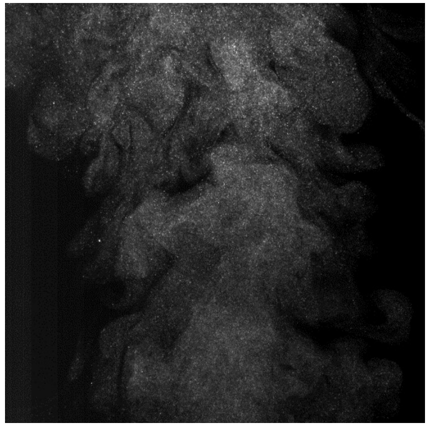
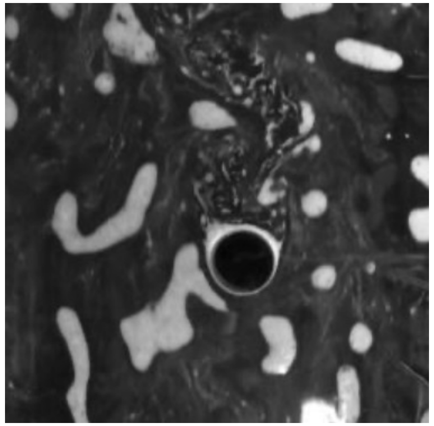
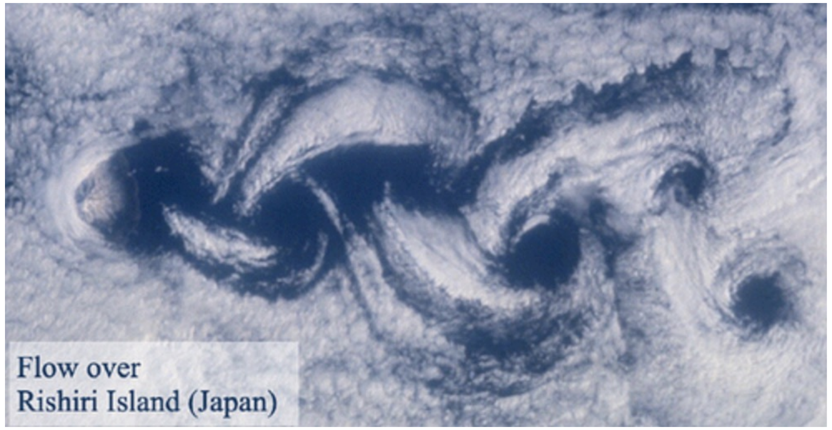
The problem of fluid motion estimation

- Fluid flow motion estimation is a topic of interest for many science and engineering fields.







$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}$$

Left: 3rd PIV Challenge Right: photograph from NASA, 2001; STS-100

Fluid Flow and Optical Flow

- Advection-Diffusion Equation

$$\frac{\partial I}{\partial t} + \nabla \cdot (I\boldsymbol{u}) = D\nabla^2 I \quad \& \quad \nabla \cdot \boldsymbol{u} = 0 \quad \& \quad D = 0$$

- I: A scalar field to be transport.
- D: Diffusion coefficient.

- Divergence-free condition

- No diffusion

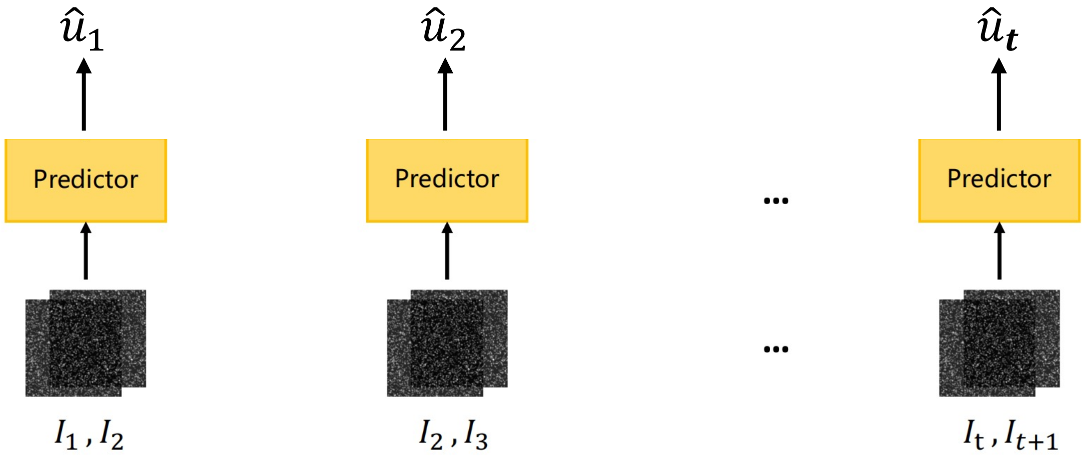
$$\boxed{\frac{\partial I}{\partial t} + \boldsymbol{u} \cdot \nabla I = 0}$$

Brightness Constancy Assumption (BCA)[1]

[1]Horn, B. and Schunck., B. Determining optical flow. Artificial Intelligence, 17:185–203, 1981.

Motivation

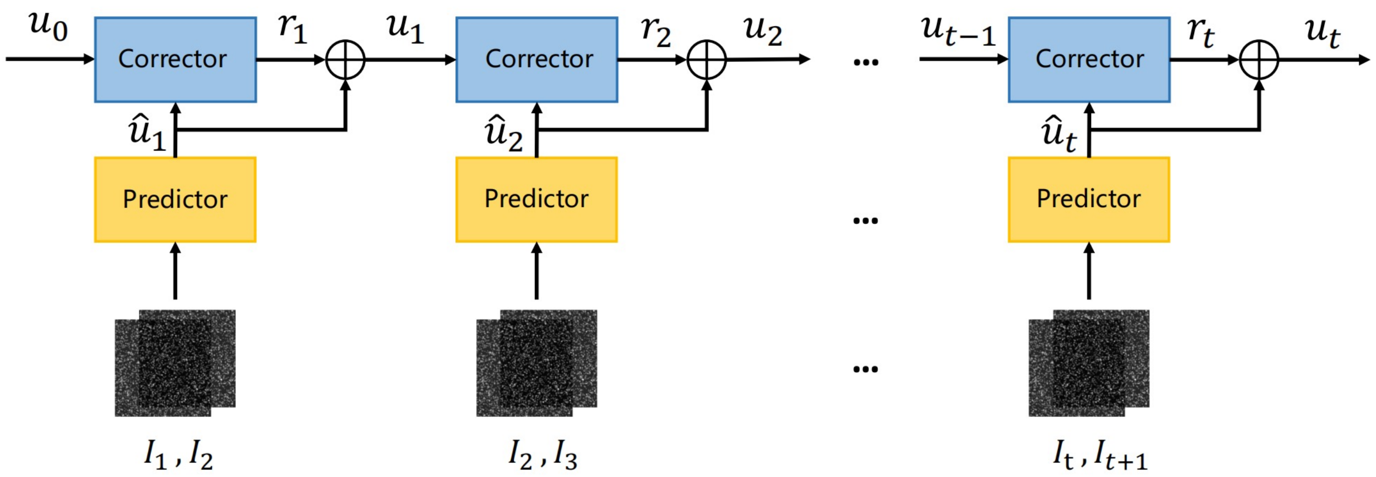
- An approach based on BCA alone misses key dynamics when applied to fluid flow



- Given a sequence of consecutive fluid observation images $\boldsymbol{I} = (I_1, I_2, I_3, \dots, I_T)$
- To estimate the dense flow field (displacement field)

Prediction and Correction

- Design an optical flow based predictor constrained by PDE
- Use a fluid informed corrector to compensate the missing dynamics



- Given a sequence of consecutive fluid observation images $\boldsymbol{I} = (I_1, I_2, I_3, \dots, I_T)$
- To estimate the dense flow field (displacement field)

Fluid Motion Predictor

- Variational optical flow approach constrained by the **Stokes equation**.

$$\min_{\boldsymbol{u}} \int \underbrace{\left(\frac{\partial I}{\partial t} + \boldsymbol{u} \cdot \nabla I\right)}_{\text{Data term}} + \underbrace{\mu|\nabla \boldsymbol{u}|^2}_{\text{Smoothness}} + \underbrace{p\nabla \cdot \boldsymbol{u}}_{\text{Divergence}} \, d\boldsymbol{x}$$

Fluid Motion Predictor

- Variational optical flow approach constrained by the **Stokes equation**.

$$\min_{\boldsymbol{u}} \int \underbrace{\left(\frac{\partial I}{\partial t} + \boldsymbol{u} \cdot \nabla I\right)} + \underbrace{\mu |\nabla \boldsymbol{u}|^2} + \underbrace{p \nabla \cdot \boldsymbol{u}}\, d\boldsymbol{x}$$

Its Euler-Lagrange equation is Stokes equation[1]:

$$\underbrace{-\mu \nabla^2 \boldsymbol{u}} + \underbrace{\nabla p} = \underbrace{\nabla I}$$

[1]Note: The derivation is shown in Appendix B of the paper.

Physical Corrector

- Generalise to a generic operator splitting scheme

$$\frac{\boldsymbol{u}_t^* - \boldsymbol{u}_{t-1}}{\Delta t} = \boldsymbol{u}_{t-1} \cdot \nabla \boldsymbol{u}_{t-1} + \nu \nabla^2 \boldsymbol{u}_{t-1}$$

$$\frac{\boldsymbol{u}_t - \boldsymbol{u}_t^*}{\Delta t} = -\frac{1}{\rho} \nabla p_t$$

Starting from Chorin's projection method[1]

[1]Chorin, A. J. Numerical solution of the navier-stokes equations. Math. Comp., pp. 745–762, 1968.

Physical Corrector

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Physical Corrector

- Generalise to a generic operator splitting scheme

$$\frac{\boldsymbol{u}_t - \boldsymbol{u}_{t-1}}{\Delta t} = -\frac{1}{\rho}\nabla p_t + \mathcal{R}_1$$

$$\mathcal{R}_1 = \boldsymbol{u}_{t-1} \cdot \nabla \boldsymbol{u}_{t-1} + \nu \nabla^2 \boldsymbol{u}_{t-1}$$

This recovers the incompressible Navier- Stokes equation

Physical Corrector

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Replace the pressure gradient term

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$$\mathcal{R}_1 = \boldsymbol{u}_{t-1} \cdot \nabla \boldsymbol{u}_{t-1} + \nu \nabla^2 \boldsymbol{u}_{t-1}$$

Approximate the $\boldsymbol{u}_t$ using estimated $\hat{\boldsymbol{u}}_t$ from the predictor and the tentative velocity $\boldsymbol{u}_t^*$

Physical Corrector

- Generalise to a generic operator splitting scheme

$$\frac{\boldsymbol{u}_t - \boldsymbol{u}_{t-1}}{\Delta t} = \frac{\boxed{\boldsymbol{u}_t} - \boldsymbol{u}_t^*}{\Delta t} + \mathcal{R}_1 \qquad \tilde{\boldsymbol{u}}_t = \mathrm{K}_t \odot \hat{\boldsymbol{u}}_t + (1 - \mathrm{K}_t) \odot \boldsymbol{u}_t^*$$

$$\mathcal{R}_1 = \boldsymbol{u}_{t-1} \cdot \nabla \boldsymbol{u}_{t-1} + \nu \nabla^2 \boldsymbol{u}_{t-1}$$

K_t : control factor ($0 < \mathrm{K}_t < 1$)

Approximate the $\boldsymbol{u}_t$ using estimated $\hat{\boldsymbol{u}}_t$ from the predictor and the tentative velocity $\boldsymbol{u}_t^*$

Physical Corrector

- Generalise to a generic operator splitting scheme

$$\frac{\boldsymbol{u}_t - \boldsymbol{u}_{t-1}}{\Delta t} = \frac{\widetilde{\boldsymbol{u}}_t - \boldsymbol{u}_t^*}{\Delta t} + \mathcal{R}_1 + \mathcal{R}_2$$

$$\mathcal{R}_1 = \boldsymbol{u}_{t-1} \cdot \nabla \boldsymbol{u}_{t-1} + \nu \nabla^2 \boldsymbol{u}_{t-1} \qquad \mathcal{R}_2 = \frac{\boldsymbol{u}_t - \widetilde{\boldsymbol{u}}_t}{\Delta t}$$

$\mathcal{R}_2$ is used to model the physical residual induced by the predictor and the neglection of the pressure gradient term

Physical Corrector

- Generalise to a generic operator splitting scheme

$$\frac{\boldsymbol{u}_t - \boldsymbol{u}_{t-1}}{\Delta t} = \frac{\widetilde{\boldsymbol{u}}_t - \boldsymbol{u}_t^*}{\Delta t} + \mathcal{R}_1 + \mathcal{R}_2$$

$$\mathcal{R}_1 = \boldsymbol{u}_{t-1} \cdot \nabla \boldsymbol{u}_{t-1} + \nu \nabla^2 \boldsymbol{u}_{t-1}$$

$$\boxed{\mathcal{R}_2 = \frac{\boldsymbol{u}_t - \widetilde{\boldsymbol{u}}_t}{\Delta t}}$$

To "project" the errors induced by the predictor and the velocity field's divergence in one shot

Physical Corrector

- Model the residual

$$\boldsymbol{u}_t = \widetilde{\boldsymbol{u}}_t + \Delta t \mathcal{R}_2$$

$$\boldsymbol{\Gamma}\big(\boldsymbol{\Psi}(\boldsymbol{x}, t)\big) = \sum_{i,j:i+j<q} c_{i,j} \frac{\partial^{i+j} \boldsymbol{\Psi}}{\partial x^i \partial y^j}(\boldsymbol{x}, t)$$

- The residual can be modelled by PDEs
- Use generic linear combination of partial derivatives to model PDEs

Physical Corrector

- Model the residual

$$\boldsymbol{u}_t = \tilde{\boldsymbol{u}}_t + \boxed{\Delta t \mathcal{R}_2} \qquad \boldsymbol{\Phi}(\boldsymbol{u}_{t-1}, \hat{\boldsymbol{u}}_t) = \boldsymbol{\Gamma}(\hat{\boldsymbol{u}}_t - \boldsymbol{u}_{t-1})$$

$$\boldsymbol{\Gamma}\big(\boldsymbol{\Psi}(\boldsymbol{x}, t)\big) = \sum_{i,j:i+j<q} c_{i,j} \frac{\partial^{i+j}\boldsymbol{\Psi}}{\partial x^i \partial y^j}(\boldsymbol{x}, t)$$

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Physical Corrector

- Model the residual

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- The residual can be modelled by PDEs
- Use generic linear combination of partial derivatives to model PDEs

Physical Corrector

- Prediction-Correction Scheme

$$\boldsymbol{u}_t = \widetilde{\boldsymbol{u}}_t + \boldsymbol{\Phi}(\boldsymbol{u}_{t-1}, \widehat{\boldsymbol{u}}_t)$$

$$\widetilde{\boldsymbol{u}}_t = \mathrm{K}_t \odot \widehat{\boldsymbol{u}}_t + (1 - \mathrm{K}_t) \odot \boldsymbol{u}_t^*$$

Physical Corrector

- Prediction-Correction Scheme

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$$\widetilde{\boldsymbol{u}}_t = \mathrm{K}_t \odot \widehat{\boldsymbol{u}}_t + (1 - \mathrm{K}_t) \odot \boldsymbol{u}_t^*$$

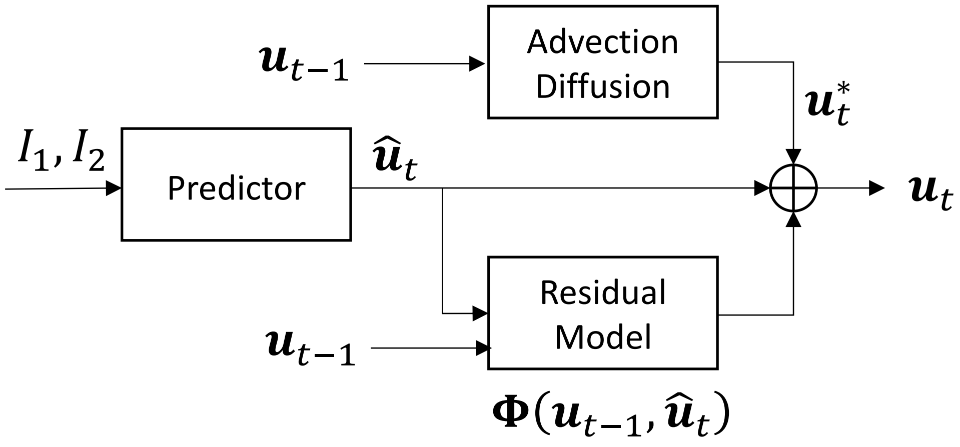
$$\boldsymbol{u}_t = \underbrace{\mathrm{K}_t \odot \widehat{\boldsymbol{u}}_t}_{\text{Predictor}} + \underbrace{(1 - \mathrm{K}_t) \odot \boldsymbol{u}_t^* + \boldsymbol{\Phi}(\boldsymbol{u}_{t-1}, \widehat{\boldsymbol{u}}_t)}_{\text{Corrector}}$$

Physical Corrector

- Prediction-Correction Scheme

$$\boldsymbol{u}_t = \tilde{\boldsymbol{u}}_t + \boldsymbol{\Phi}(\boldsymbol{u}_{t-1}, \hat{\boldsymbol{u}}_t)$$

$$\tilde{\boldsymbol{u}}_t = \mathrm{K}_t \odot \hat{\boldsymbol{u}}_t + (1 - \mathrm{K}_t) \odot \boldsymbol{u}_t^*$$



$$\boldsymbol{u}_t = \underbrace{\mathrm{K}_t \odot \hat{\boldsymbol{u}}_t}_{\text{Predictor}} + \underbrace{(1 - \mathrm{K}_t) \odot \boldsymbol{u}_t^* + \boldsymbol{\Phi}(\boldsymbol{u}_{t-1}, \hat{\boldsymbol{u}}_t)}_{\text{Corrector}}$$

Experiment

- Training and validation on synthetic dataset

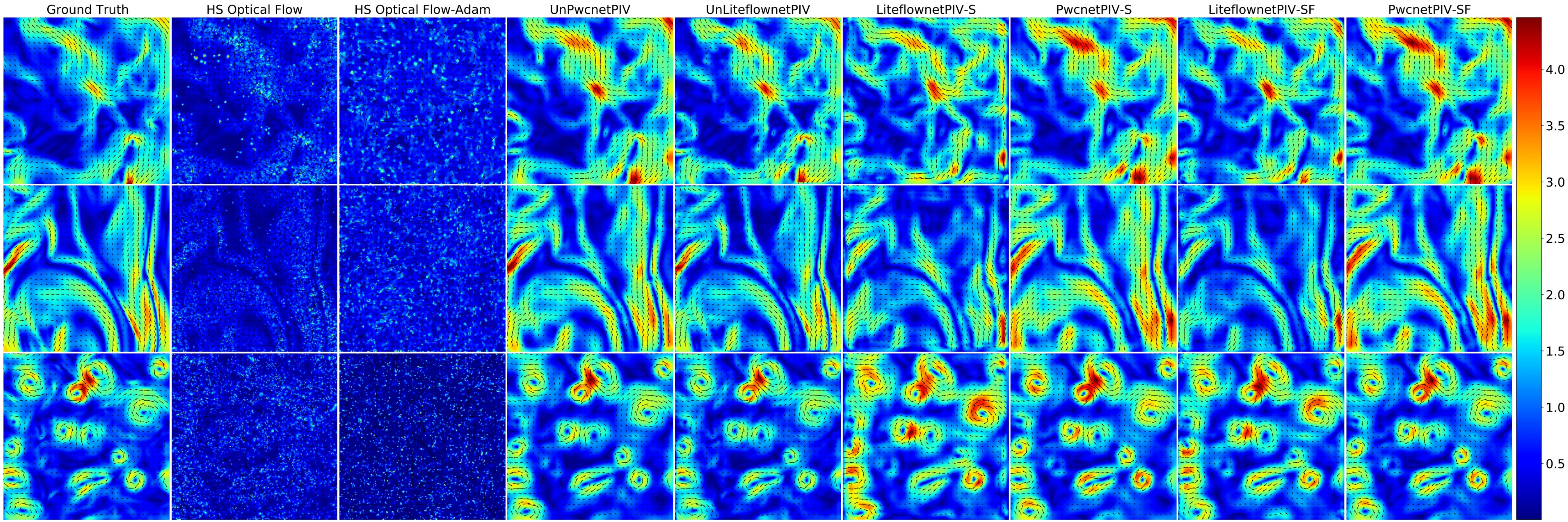
Table 2. Detailed description of the PIV dataset considered, from (Cai et al., 2019b). dx refers to the particle displacements considered between two image frames in units of number of pixels. Re refers to the Reynolds numbers considered. 'JHTDB' implies that the data was taken from the Johns Hopkins turbulence databases (Li et al., 2008). Refer to (Cai et al., 2019b) for further details.

Type Name	Description	Condition	Quantity
Uniform	Uniform flow	$\lvert dx \rvert \in [0, 5]$	1000
Back-step	Flow past a backward facing step	Re = 800	600
		Re = 1000	600
		Re = 1200	1000
		Re = 1500	1000
Cylinder	Flow past a circular cylinder	Re = 40	50
		Re = 150	500
		Re = 200	500
		Re = 300	500
		Re = 400	500
DNS-turbulence	Homogeneous and isotropic turbulent flow	-	2000
SQG	Sea surface flow driven by SQG model	-	1500
Channel flow	Channel flow provided by JHTDB	-	1600
JHTDB-mhd1024	Forced MHD turbulence provided by JHTDB	-	800
JHTDB-isotropic1024	Forced isotropic turbulence provided by JHTDB	-	2000

Methods	Back-Step		Cylinder		JHTDB channel		DNS turbulence		SQG	
	AEPE	AAE	AEPE	AAE	AEPE	AAE	AEPE	AAE	AEPE	AAE
HS Optical Flow	221.1	78.7	88.7	47.3	39.4	34.2	93.7	58.7	116.1	70.0
HS Optical Flow-Adam	200.9	62.6	58.1	25.5	16.1	13.5	57.7	30.4	70.3	35.2
LiteFlowNet-PIV-S	16.4	8.3	15.5	6.8	22.4	18.9	37.6	21.3	44.4	23.5
PwcNet-PIV-S	6.2	3.0	4.9	2.2	13.9	11.7	18.7	10.9	25.0	13.3
LiteFlowNet-PIV-SF	13.7	6.9	13.0	5.6	19.2	16.2	30.8	17.6	36.0	18.9
PwcNet-PIV-SF	**4.7**	**2.4**	**4.1**	**1.8**	12.9	10.8	16.0	9.4	22.1	11.8
UnLiteFlowNet-PIV	9.4	4.0	6.9	3.8	**8.4**	**3.3**	**15.0**	**8.6**	**17.3**	**9.0**
UnPwcNet-PIV	8.2	3.9	7.1	3.9	13.4	11.3	21.5	12.8	25.2	13.5

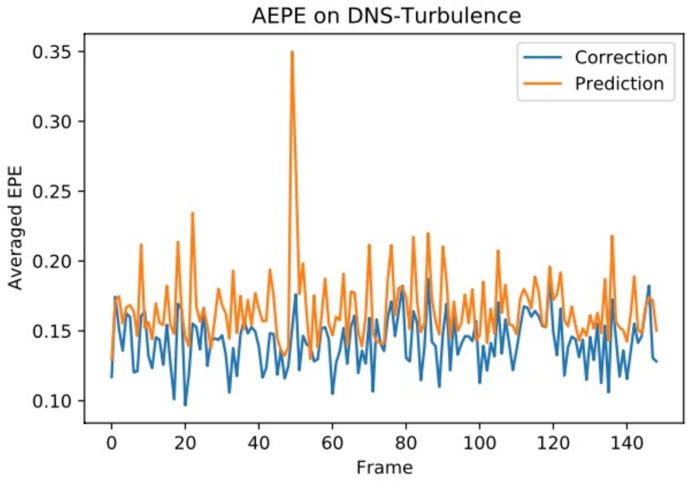
Experiment

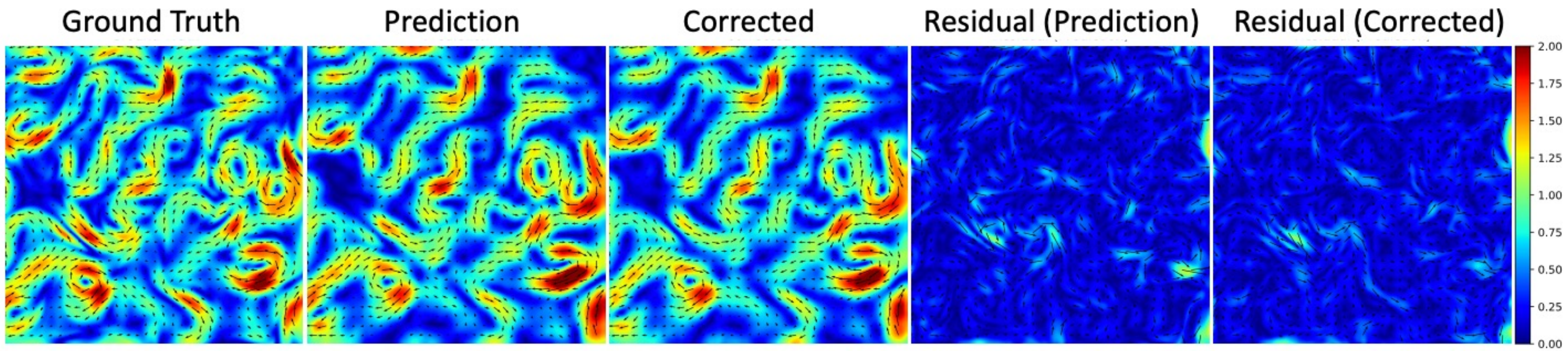
- Synthetic Dataset Visual Results



Experiment

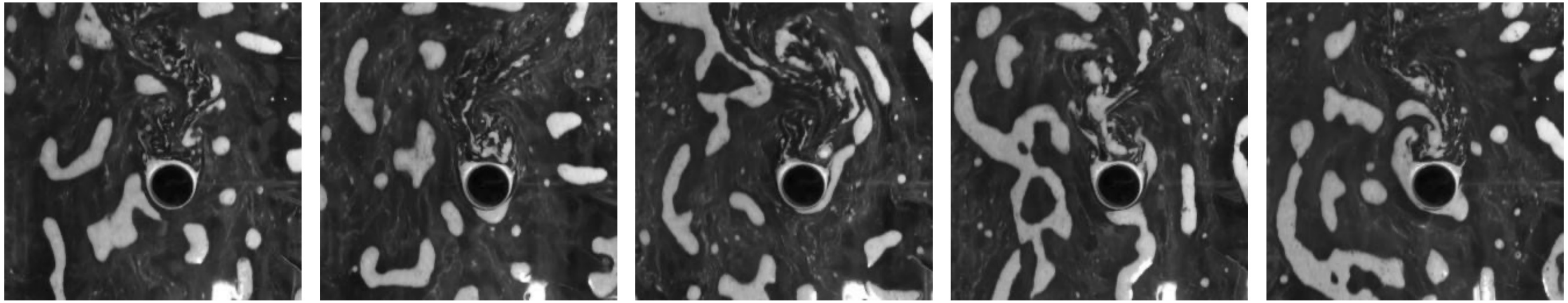
- Comparison between prediction-only and the corrected velocities



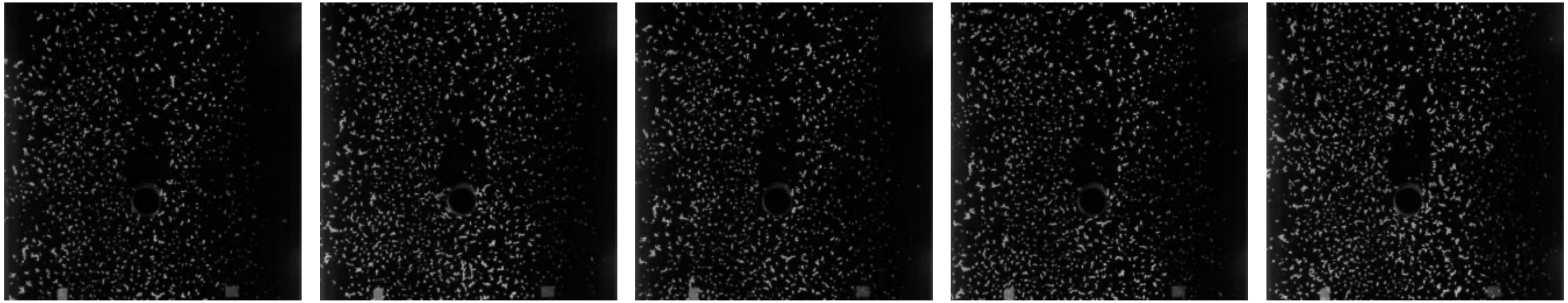


Experiment

- Test on Real world Dataset Samples



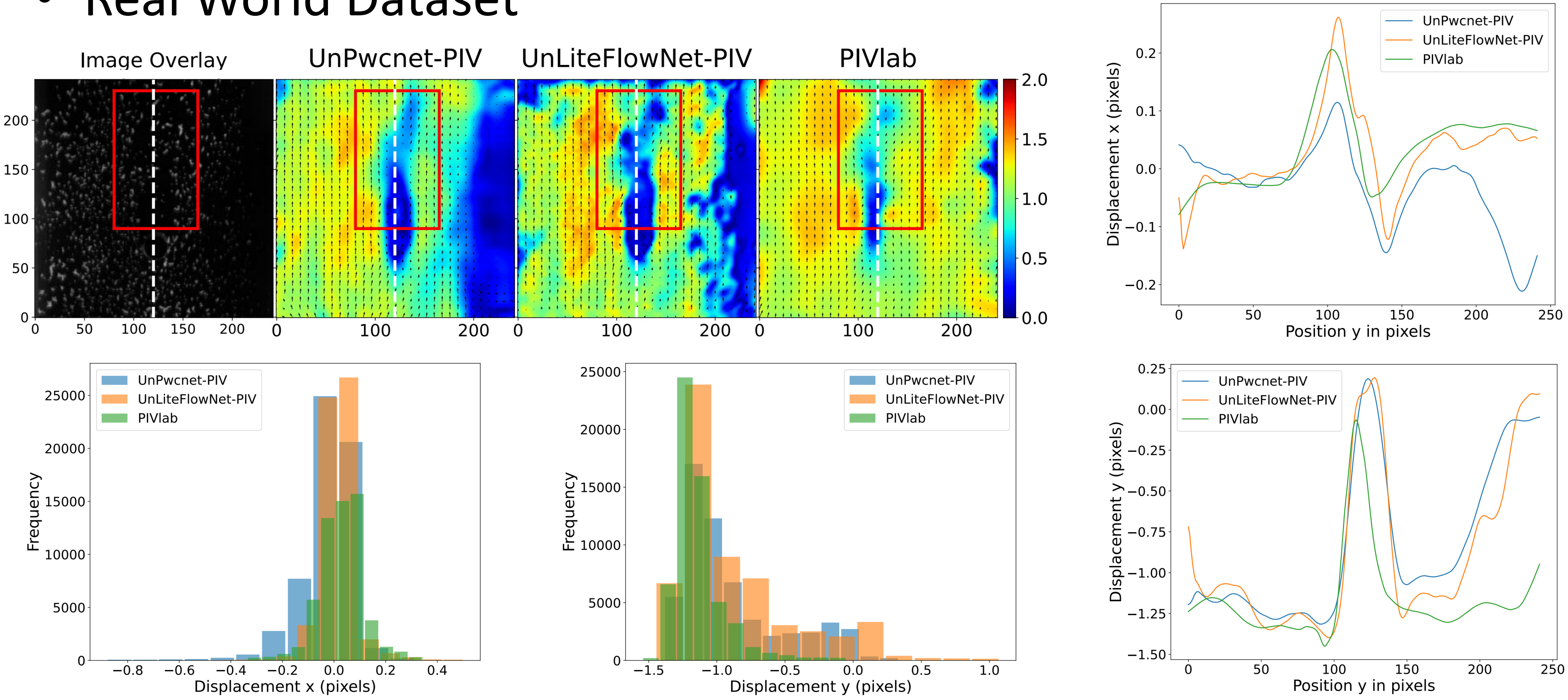
Foam



Confetti

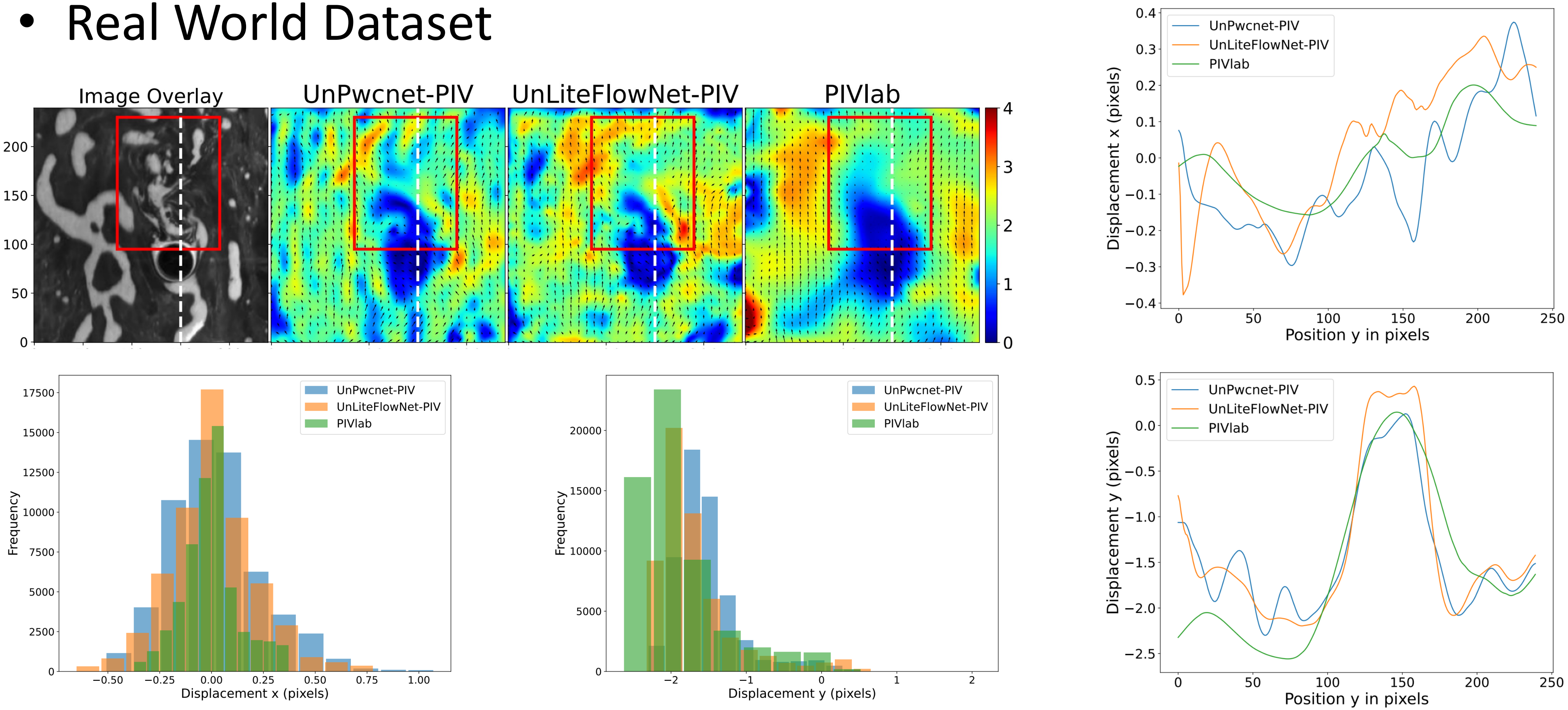
Experiment

- Real World Dataset



Experiment

- Real World Dataset



Thanks!