Robust Group Synchronization via Quadratic Programming

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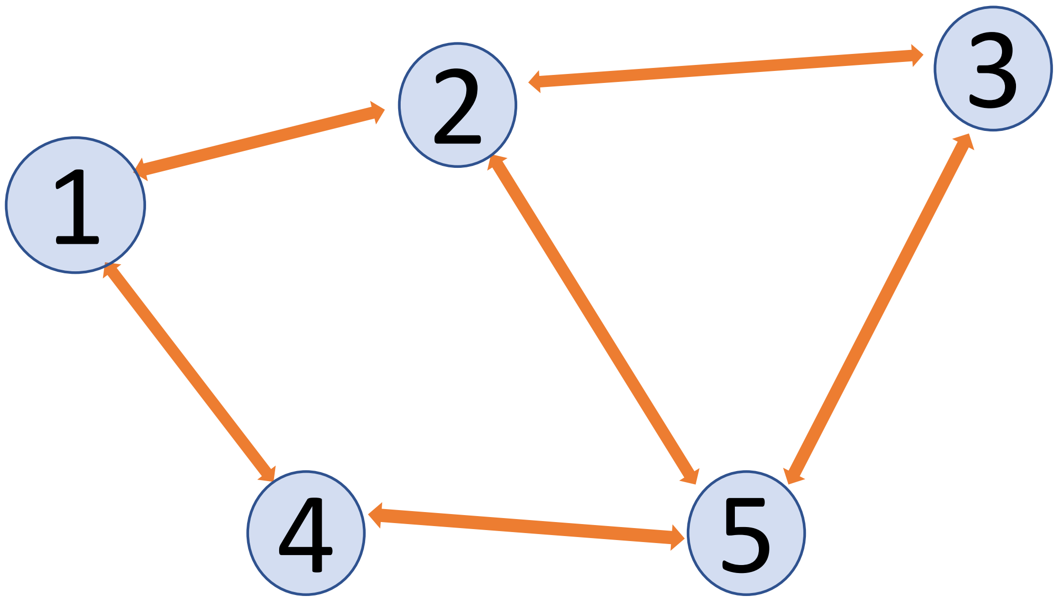
Group Synchronization (GS)

- Assumes a mathematical group $\mathcal{G}$
- Examples: 3-D rotations (SO(3)), permutation group (S_n)

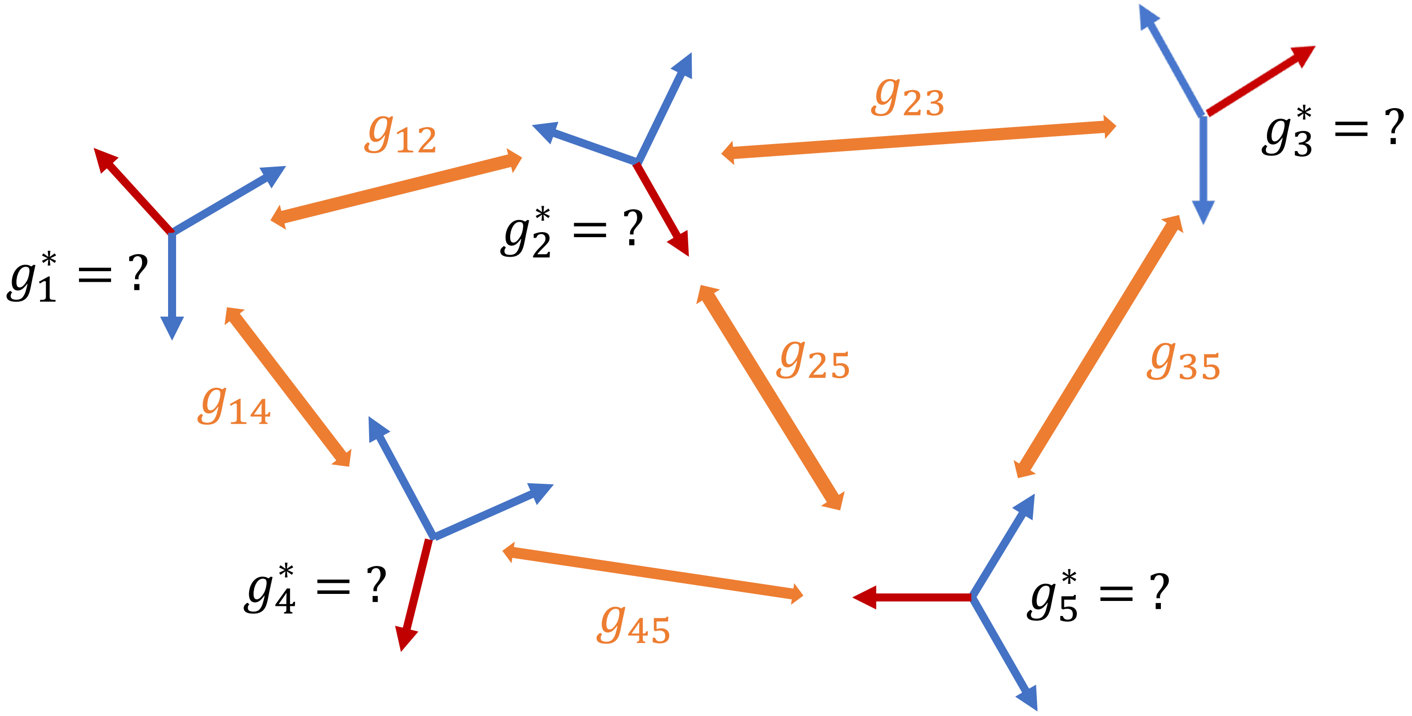
Group Synchronization



Given a graph $G([n], E)$

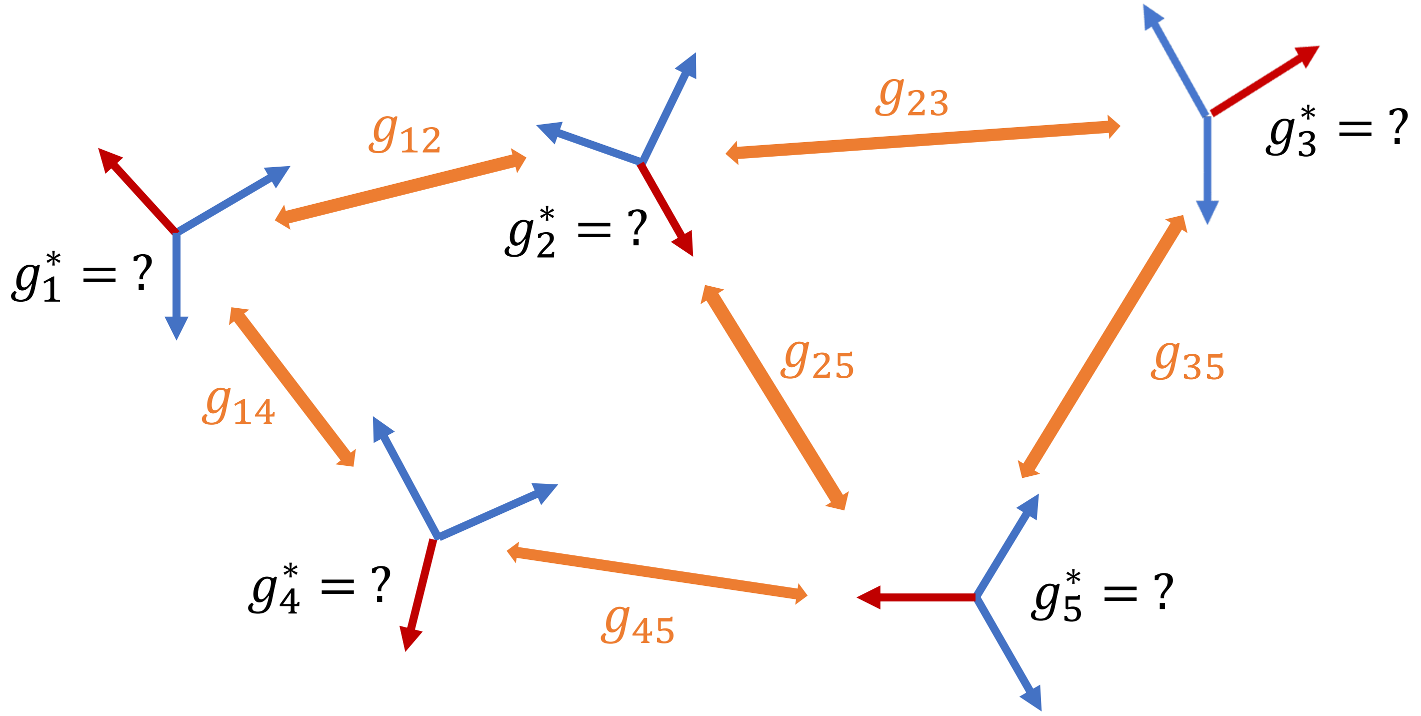
$[n] := \{1,2,3, \dots, n\}$, E is the set of edges

Group Synchronization



Each node $i \in [n]$ is assigned an *unknown* ground truth group element g_i^*

Group Synchronization



- Each edge $ij \in E$ is given a possibly *noisy* and *corrupted* group ratio g_{ij}
- The *clean* group ratio for $ij \in E$ is $g_{ij}^* = g_i^* g_j^{*-1}$
- **Group Synchronization**: Estimate $\{g_i^*\}_{i\in[n]}$ from $\{g_{ij}\}_{ij\in E}$
- $\{g_i^* g_0\}_{i\in[n]}$ for any group element g_0 is also a solution

Applications of GS

- $\mathcal{G} = \mathrm{U}(1)$: phase synchronization (cryo-EM)
- $\mathcal{G} = \mathrm{SO}(3)$: rotation averaging (SfM, SLAM)
- $\mathcal{G} = \mathrm{S}_n$: multi-image matching (SfM)
- $\mathcal{G} = \mathrm{Z}_2$: correlation clustering (community detection)

Goal of Our Work

- Reliably estimate the edge corruption levels
- Robustly estimate group elements

Corruption Estimation

$$\text{recall } g_{ij} = \begin{cases} g^*_{ij} := g^*_i g^{*-1}_j, & ij \text{ is clean} \\ \tilde{g}_{ij}, & ij \text{ is corrupted} \end{cases}$$

Goal: estimate the corruption Levels

$$s^*_{ij} := d(g_{ij}, g^*_{ij})$$

from 3-cycle inconsistencies

$$d_{ijk} := d(e, g_{ij} g_{jk} g_{ki})$$

where d is a bi-invariant distance on $\mathcal{G}$:

$$d(g_1, g_2) = d(g_1 g_3, g_2 g_3) = d(g_3 g_1, g_3 g_2)$$

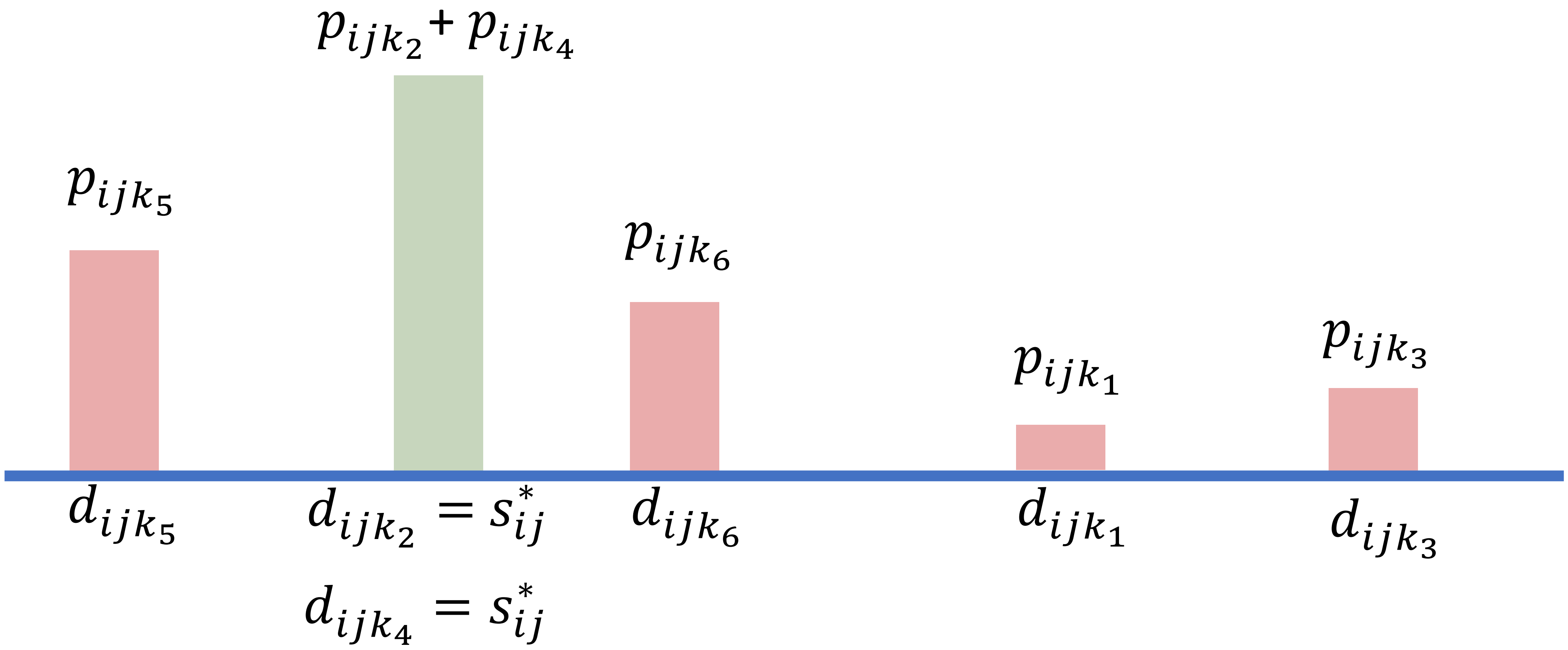
Corruption Estimation

When ik and jk are both clean, then d_{ijk} is an exact estimator of s_{ij}^*.

In such a case, we say ijk is a good cycle of edge ij

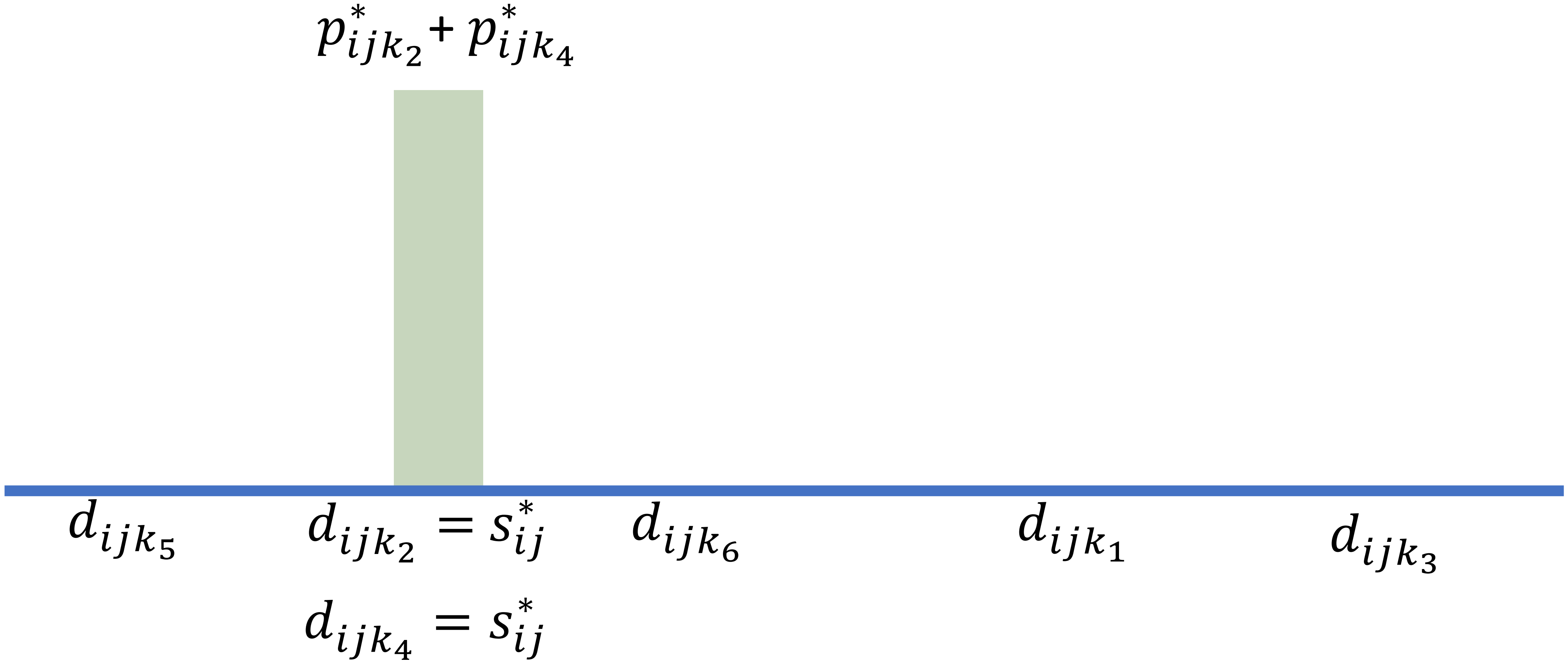
Detection and Estimation of Structural Consistency (DESC)

If $ij \in E$ is contained in at least one good cycle, then s_{ij}^* is supported on the set of d_{ijk}'s



Detection and Estimation of Structural Consistency (DESC)

The true value of p_{ijk}'s, denoted by p^*_{ijk}'s, are only nonzero at good cycles, so that the corresponding distribution concentrates on s^*_{ij}.



Detection and Estimation of Structural Consistency (DESC)

Define three vectors such that

$\boldsymbol{p}^*_{ij}(k) = p^*_{ijk}$, $\boldsymbol{v}^*_{ij}(k) = s^*_{ik} + s^*_{jk}$, and $\boldsymbol{d}_{ij}(k) = d_{ijk}$.

For any $ij \in E$, $\boldsymbol{p}^{*T}_{ij}\, \boldsymbol{d}_{ij} = s^*_{ij}$ and $\boldsymbol{p}^{*T}_{ij}\, \boldsymbol{v}^*_{ij} = 0$.

$\boldsymbol{p}^*_{ij}$ and s^*_{ij} are estimated simultaneously using above constraints.

Detection and Estimation of Structural Consistency (DESC)

Recall the two constraints $\boldsymbol{p}_{ij}^{*T}\,\boldsymbol{d}_{ij} = s_{ij}^{*}$ and $\boldsymbol{p}_{ij}^{*T}\,\boldsymbol{v}_{ij}^{*} = 0$

where $\boldsymbol{p}_{ij}^{*}(k) = p_{ijk}^{*},\ \boldsymbol{v}_{ij}^{*}(k) = s_{ik}^{*} + s_{jk}^{*}$, and $\boldsymbol{d}_{ij}(k) = d_{ijk}$

We solve the following quadratic programming problem

$$\underset{\boldsymbol{p}_{ij},\, s_{ij}}{\text{minimize}}\ \sum_{ij\in E} \boldsymbol{p}_{ij}^{T}\,\boldsymbol{v}_{ij}$$

$$\text{subject to}\quad s_{ij} = \boldsymbol{p}_{ij}^{T}\,\boldsymbol{d}_{ij}\quad \text{for } ij \in E$$

where each $\boldsymbol{p}_{ij}$ lies in a probability simplex (a linear constraint).

Detection and Estimation of Structural Consistency (DESC)

- Our objective function is an approximate upper bound of the cumulative error of the corruption estimation

- Under a mild deterministic condition, any global minimum of our DESC formulation exactly recovers the ground truth s_{ij}^*

- Given a Lie group, under the uniform corruption model (edges are i.i.d corrupted with probability q, and the corrupted group ratios follow Haar measure), the sample complexity for exact recovery of DESC is $n/\log(n) = \Omega(q^{-2})$ which matches the information-theoretic bound

DESC for Rotation Averaging

- Solve the QP formulation by a projected gradient descent (using Riemannian gradient)

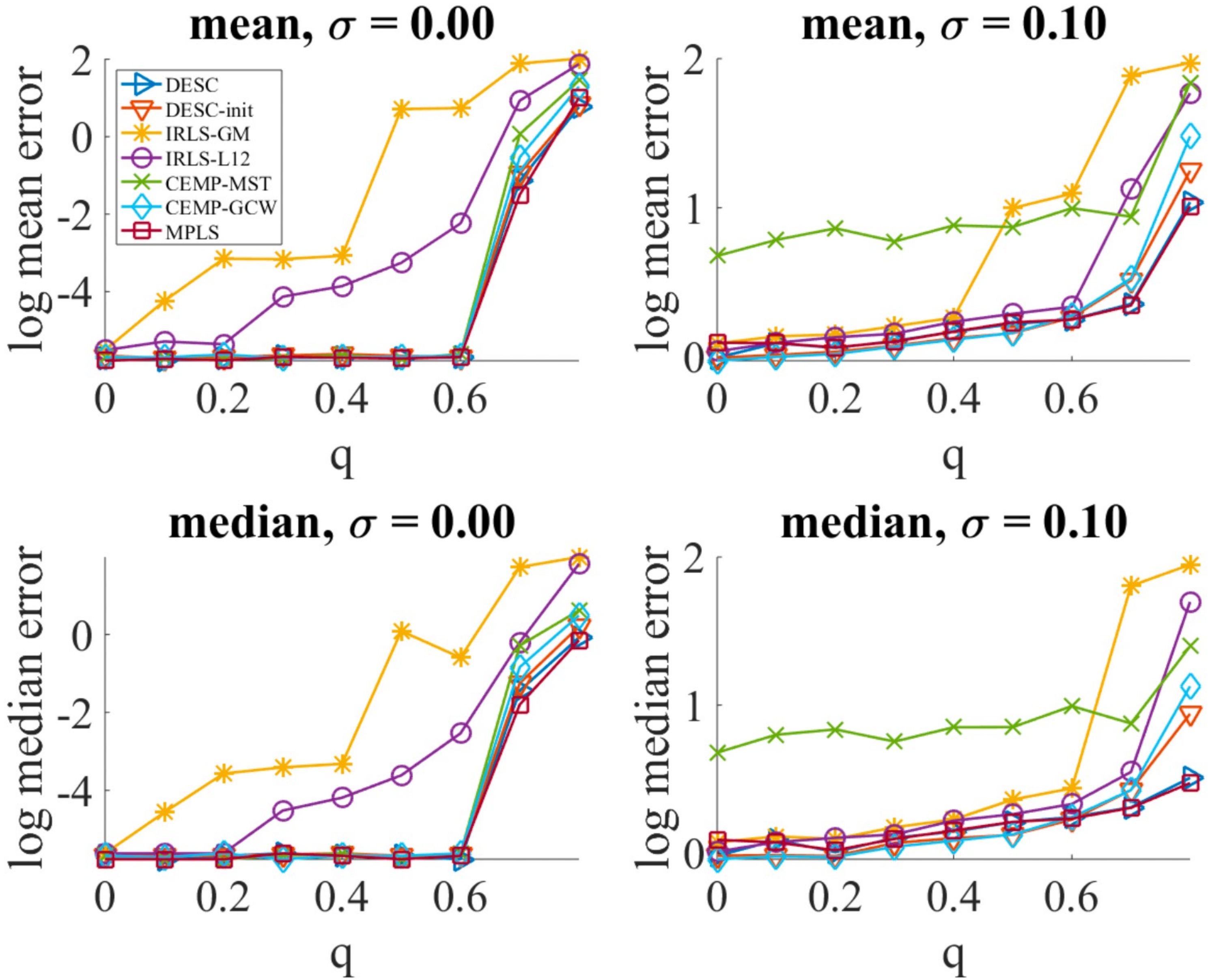
- After solving s_{ij}, approximately solve a weighted least squares for group elements by a spectral method, where edge weights $w_{ij} = s_{ij}^{-3/2}$.

- One can further refine the initialized solution by a modified iteratively reweighted least squares (IRLS), where the residuals are adjusted by the DESC-estimated corruption levels.

Synthetic Data Experiments

Uniform Corruption Model (UCM):

- Edges are i.i.d corrupted with probability q
- Additive noise with noise level σ for all edges
- The corrupted group ratios follow Haar measure

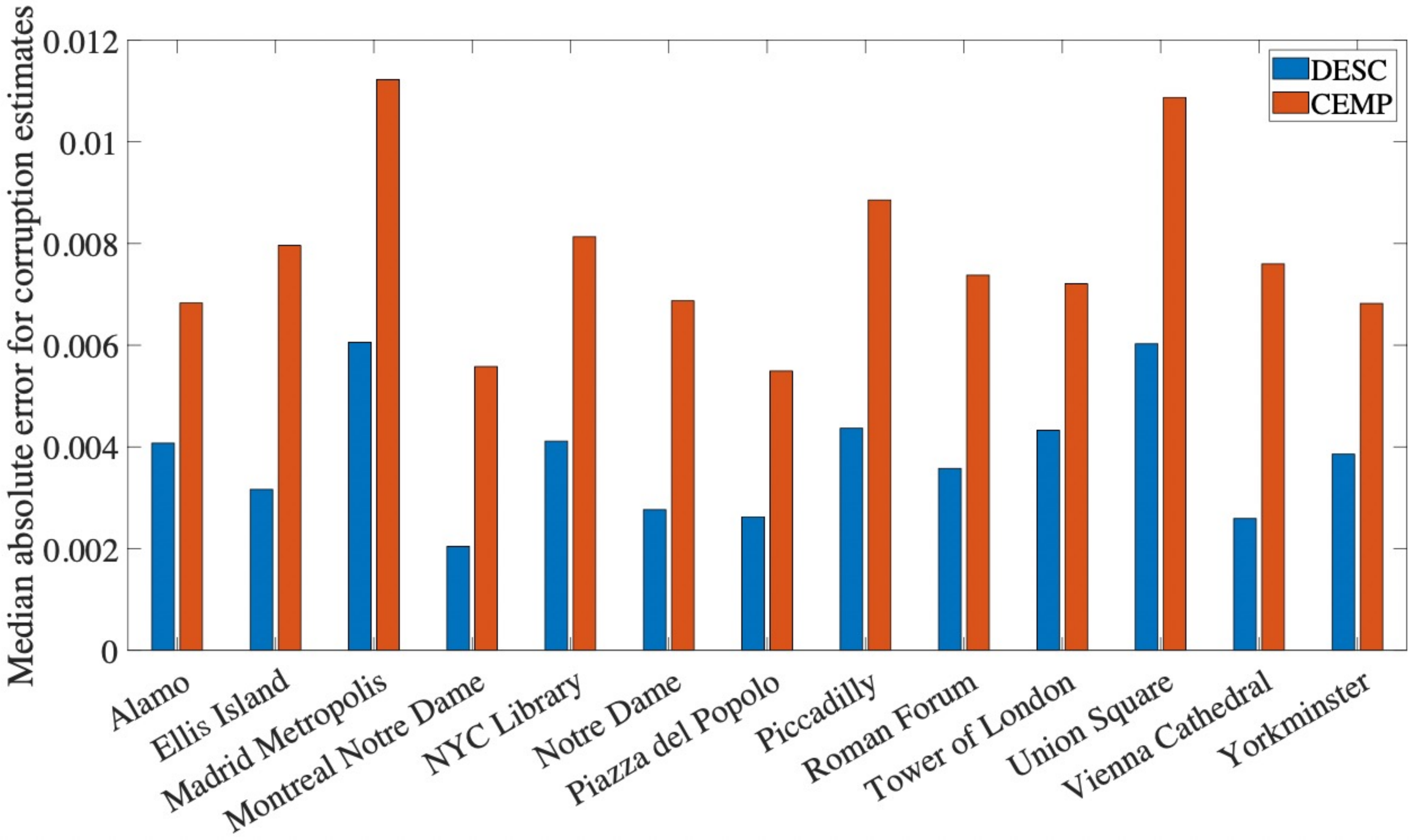

mean, $\sigma = 0.00$
mean, $\sigma = 0.10$
median, $\sigma = 0.00$
median, $\sigma = 0.10$
log mean error
log median error
q
DESC
DESC-init
IRLS-GM
IRLS-L12
CEMP-MST
CEMP-GCW
MPLS

Real Data Experiments

- 13 Photo Tourism Datasets
- Camera relative rotations are estimated by the LUD pipeline (özyeşil and Singer 2015)

Table 1. Average of the mean and median errors (in degrees) for rotation estimates across the 13 datasets of Photo Tourism

	DESC	DESC-init	IRLS-GM	IRLS-$L_{\frac{1}{2}}$	CEMP-MST	CEMP-GCW	MPLS
mean	**3.5119**	3.8354	3.9644	3.8447	4.1447	3.9191	3.7142
median	**1.5938**	1.8516	1.7255	1.7201	1.7975	2.0339	1.7032



Conclusion

- We proposed the DESC framework for robustly solving GS problem
- Our QP formulation has clear interpretation and enjoys theoretical guarantees
- Experiments show superior performance of our method on rotation averaging

Future directions:

- Study the optimal ways of assigning edge weights
- Extend the idea DESC framework to other tasks with structural consistency, such as subspace recovery and rank aggregation
- Generalize DESC to incorporate longer cycles in order to handle sparser graphs