Hardness and Algorithms for Sparse Optimization

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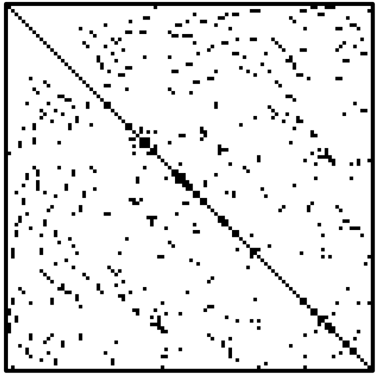
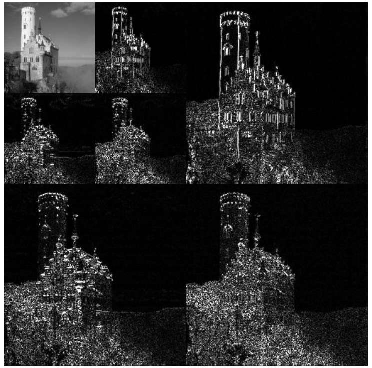
What is this paper about?

- ➢ Sparsity in data is a natural pattern
- ➢ Study optimization problems with sparsity constraints





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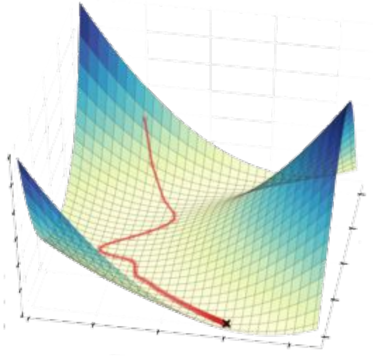

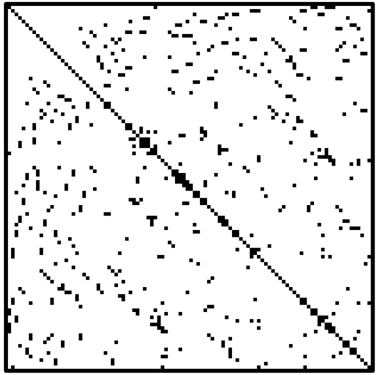

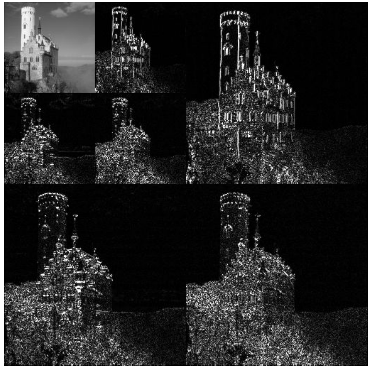

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- ➢ Unified approach to upper bound algorithms for sparse optimization problems

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Very related problems: We show robust regression reduces to sparse regression!

A common algorithmic approach

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- ➢ Consider sparse regression: "Find a k-sparse x to solve Ax = b"
- ➢ Suppose there exists y with $\|y\|_0 = k$ with Ay = b (**Realizable case**)

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Brute force algorithm: Guess support of y: roughly **d^k time** where A in $R^{n \times d}$

Prior work: work very hard to get roughly ~ **d^{k-1} time** (Har-Peled, Indyk, Mahabadi ‘18)

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Our result: ~ **$d^{k/2}$** time.

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(1) Enumerate over all k/2 sparse z and compute Az

(2) Check if there exists k/2 sparse z' such that Az' = b-Az

(3) Output z'+z as solution

Only ~ $\mathbf{d^{k/2}}$ cost!

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(1) Enumerate over all k/2 sparse z and compute Az ← Enumerate over net

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(1) Enumerate over all k/2 sparse z and compute Az ← Enumerate over net

(2) Check if there exists k/2 sparse z' such that Az' = b-Az ← Check for z' using nearest neighbor data structure on {Az}

(3) Output z'+z as solution

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Theorem 1.4. *Given* $A \in \mathbb{R}^{n \times d}$ *and* $b \in \mathbb{R}^n$ *such that there exists a* k*-sparse vector* x *satisfying* $Ax = b$*, there exists an algorithm that returns a* k*-sparse vector* $z \in \mathbb{R}^d$ *satisfying* $\|Az - b\|_2 \le \varepsilon$ *in time*

$$\min_{c \ge 1} O\left(nk \cdot \left(\frac{12c \cdot d \cdot \|b\|_2}{\varepsilon} \right)^{\frac{k}{2} \cdot (1 + 1/(2c^2 - 1))} \right).$$

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Theorem 1.4. *Given $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ such that there exists a k-sparse vector x satisfying $Ax = b$, there exists an algorithm that returns a k-sparse vector $z \in \mathbb{R}^d$ satisfying $\|Az - b\|_2 \leq \varepsilon$ in time*

$$\min_{c \geq 1} O\left(nk \cdot \left(\frac{12c \cdot d \cdot \|b\|_2}{\varepsilon}\right)^{\frac{k}{2} \cdot (1 + 1/(2c^2 - 1))}\right).$$

Similar improvement for robust regression ($\mathbf{n^k \rightarrow n^{k/2}}$)

Other results

Upper bounds:

- Sparse PCA problem: “Find a k sparse unit vector u to maximize $u^T A u$”

Lower bounds side:

- Bicriteria hardness for sparse regression: “Problem still hard even if we relax sparsity and approximation constraints”
- Fine grained hardness for robust regression: Evidence of $\mathbf{n^{k/2}}$ hardness for robust regression

Thank you!

Please see our paper for full details!