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On the Convergence of the Shapley Value in Parametric Bayesian Learning Games

Lucas Agussurja Xinyi Xu Bryan Kian Hsiang Low

Introduction

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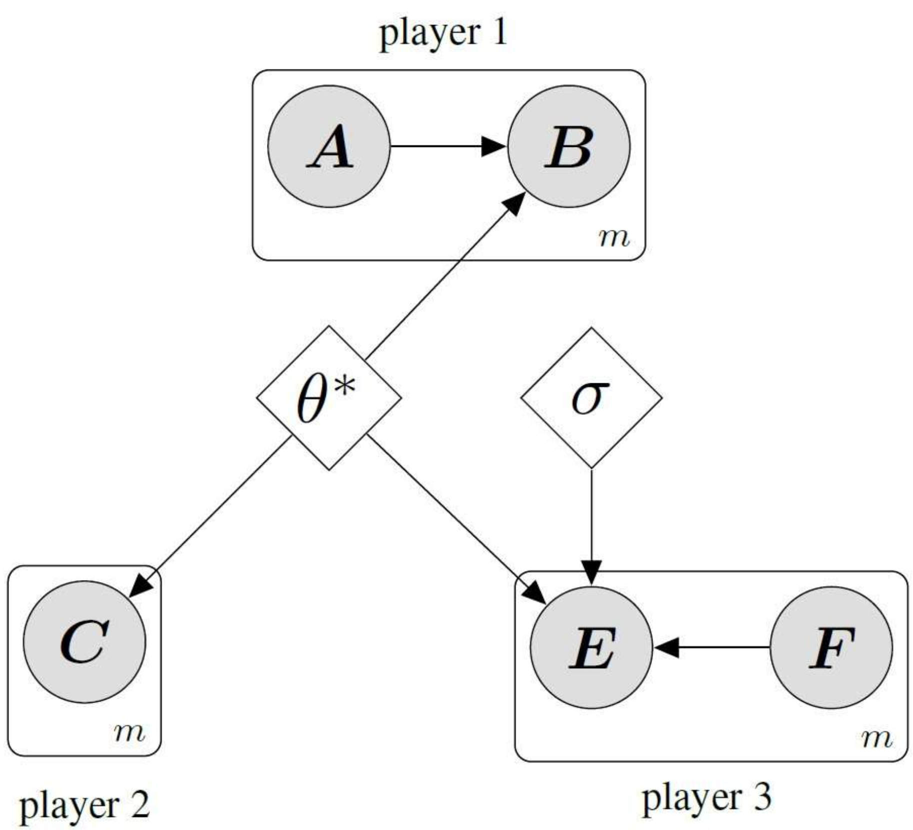
- Measuring datasets **contributions** in a **joint Bayesian inference**
- **Cooperative game theory** perspective
- Characteristic function: **Posterior-prior information gain**
- **Convergence** properties of the **Shapley value**

Collaborative Setting: Example



Collaborative Setting: Inference

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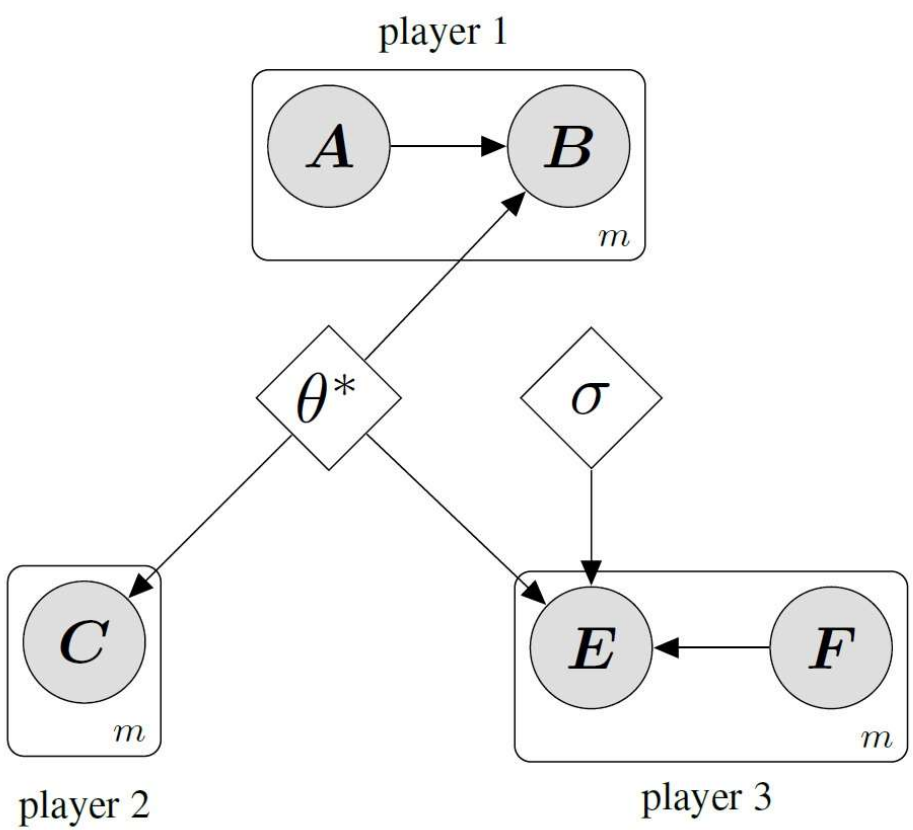

Joint Inference:

$$\boldsymbol{p}_S^m(\theta) = \frac{\mathcal{L}_S(\theta; \mathbf{X}_S^m)\ \pi(\theta)}{\int_{\theta' \in \Theta} \mathcal{L}_S(\theta'; \mathbf{X}_S^m)\ \mathrm{d}\Pi(\theta')}$$

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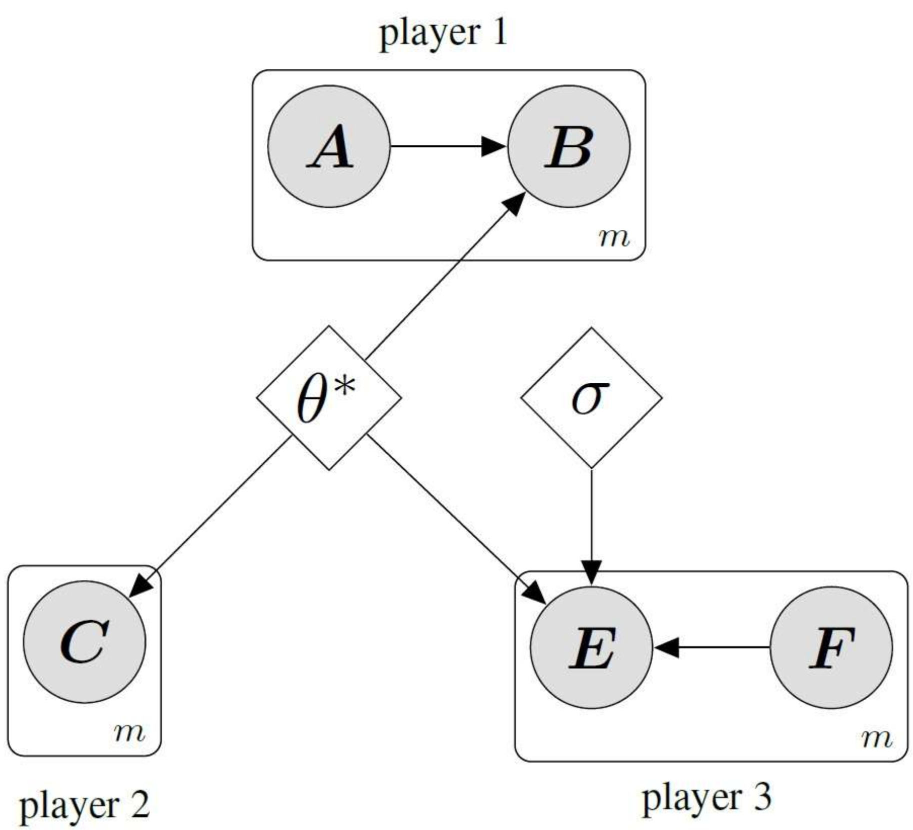

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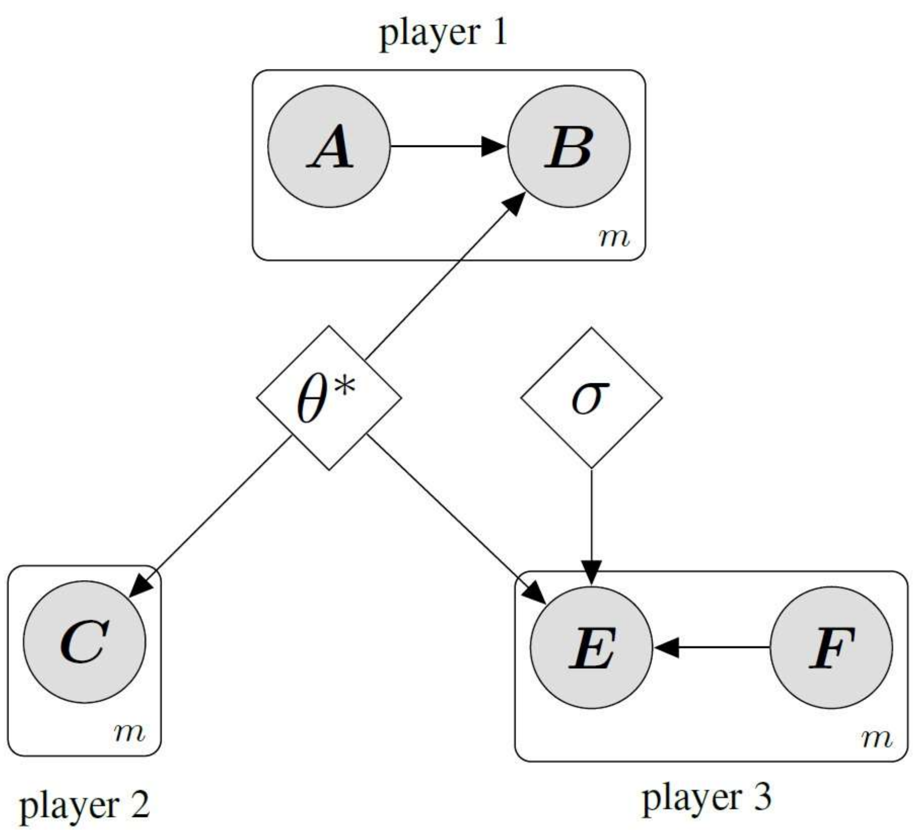

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Collaborative Setting: Assumptions

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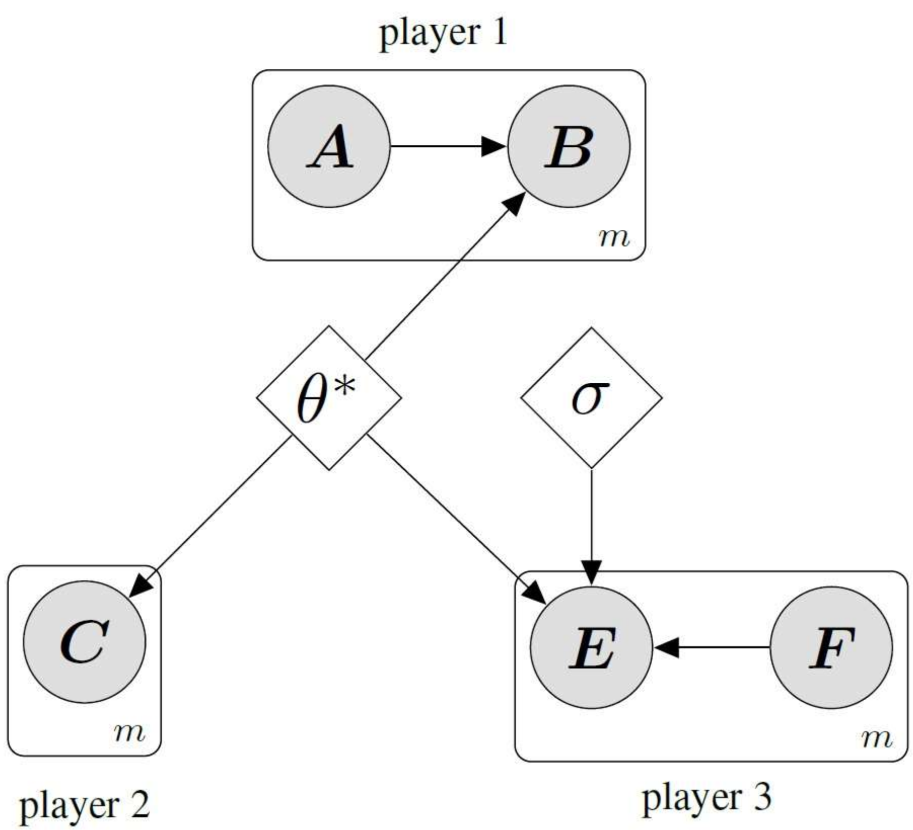

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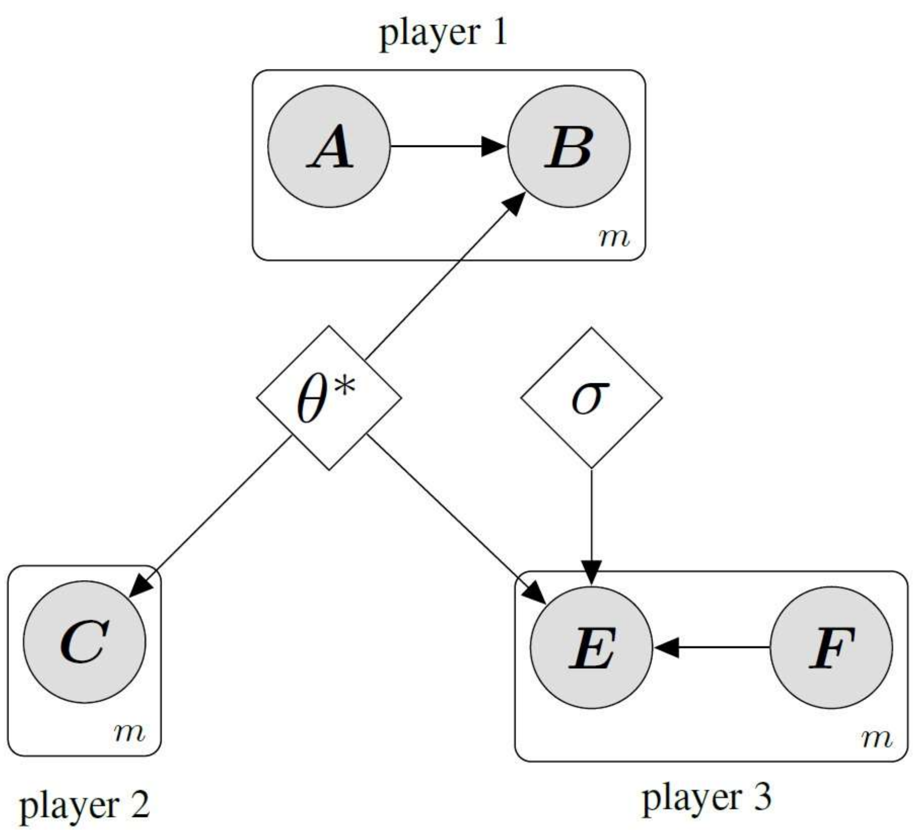
Assumption 1: Conditional Independence

$$\mathcal{L}_S(\theta; \mathbf{X}_S^m) = \prod_{i \in S} \mathcal{L}_i(\theta; \mathbf{X}_i^m)$$

Collaborative Setting: Assumptions



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Assumption 2: Twice Differentiability

$$\mathcal{L}_S(\theta; \mathbf{X}_S^m) = \prod_{i \in S} \mathcal{L}_i(\theta; \mathbf{X}_i^m)$$

Collaborative Setting: Assumptions

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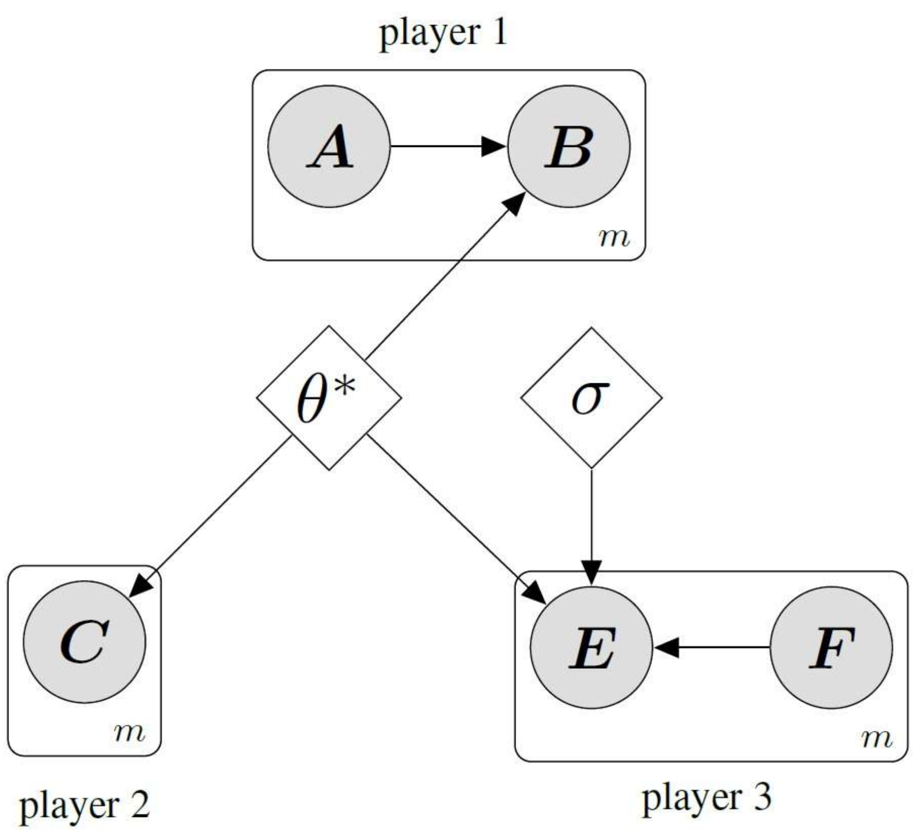

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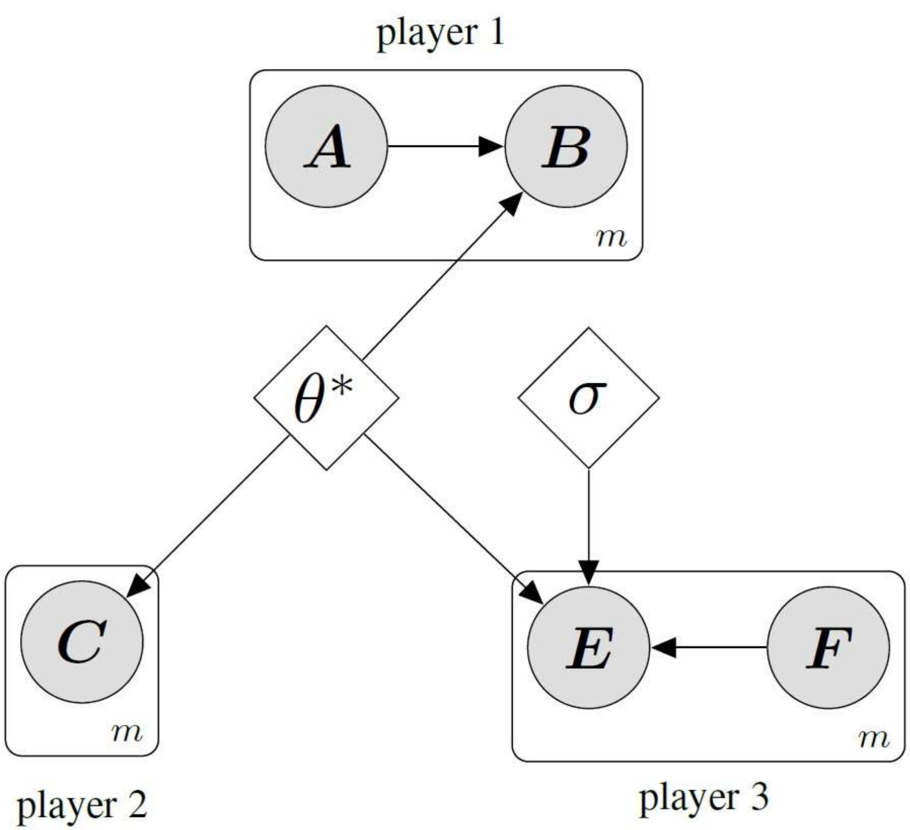
Assumption 3:
Distinguisability

$$\mathcal{L}_S(\theta; \mathbf{X}_S^m) = \prod_{i \in S} \mathcal{L}_i(\theta; \mathbf{X}_i^m)$$

Collaborative Setting: Assumptions



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Assumption 4: The Prior supports the true parameter

Cooperative Game Formulation

- **Characteristic function**:

$$\boldsymbol{\mathcal{V}}^m(S) := \mathrm{KL}(\boldsymbol{P}_S^m \| \Pi) := \int_{\Theta} \log\left(\mathrm{d}\boldsymbol{P}_S^m / \mathrm{d}\Pi\right) \mathrm{d}\boldsymbol{P}_S^m$$

- **Shapley value**:

$$\phi(i; \boldsymbol{\mathcal{V}}^m) := \sum_{S \subseteq N \setminus \{i\}} w_S \left[\boldsymbol{\mathcal{V}}^m(S \cup \{i\}) - \boldsymbol{\mathcal{V}}^m(S)\right]$$

Main Theoretical Result

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- Define the **limiting game**:

$$V(S) := 0.5 \log |\mathcal{I}_S| \quad \text{if } S \text{ is nonempty,}$$
$$V(S) := 0 \quad \text{otherwise}$$

- Under **Assumptions 1-4**
- For a **uniformly* distributed prior**
- The following holds:

$$\phi(i; \boldsymbol{\mathcal{V}}^m) - \phi(j; \boldsymbol{\mathcal{V}}^m) \xrightarrow{\text{p}} \phi(i; V) - \phi(j; V)$$

for any two players i and j.

High Level Proof Ideas

- Assumptions 1-4 allow for **closed form approximation** of joint posteriors
- This is given by **Bernstein-von Mises theorem**
- Approximate KL divergence using the **normal posterior**
- Characteristic function **decomposes**
- By **linearity** of Shapley value, when taking differences:
  - Constant terms cancel out
  - Left with term that depends only on Fisher information

Collaborative Online Framework

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- **Bundling of data points:**

$$\mathrm{Var}\left[\frac{\partial}{\partial\theta}\log\prod_{j=1}^{r}\mathcal{L}_1(\theta^*;\mathbf{X}_{1j})\right]=\sum_{j=1}^{r}\mathrm{Var}\left[\frac{\partial}{\partial\theta}\log\mathcal{L}_1(\theta^*;\mathbf{X}_{1j})\right]=r\mathcal{I}_1$$

- **Setting:**

$$r_1 : r_2 = |\mathcal{I}_2|^{1/k} : |\mathcal{I}_1|^{1/k}$$

- **Results in:**

$$|r_1\mathcal{I}_1| = r_1^k|\mathcal{I}_1| = \left(\sqrt[k]{\frac{|\mathcal{I}_2|}{|\mathcal{I}_1|}}\,r_2\right)^k \cdot |\mathcal{I}_1| = r_2^k|\mathcal{I}_2| = |r_2\mathcal{I}_2|$$

Collaborative Online Framework

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In each iteration:

1. Use current available data points to compute an estimate $\bar{\theta}$ of the true parameter.

2. Let m_i denote the number of player i's data points received so far. For $i = 1, 2$, use sample approximation to estimate the Fisher information at $\bar{\theta}$:

$$\hat{\mathcal{I}}_i = \frac{1}{m_i} \sum_{j=1}^{m_i} \left(\frac{\partial}{\partial \theta} \log \mathcal{L}_i(\bar{\theta}; x_{ij}) \right) \left(\frac{\partial}{\partial \theta} \log \mathcal{L}_i(\bar{\theta}; x_{ij}) \right)^\top .$$

3. Collect r_1 and r_2 data points from the respective players 1 and 2 s.t. the proportion $m_1 + r_1 : m_2 + r_2$ of their cumulative data points is equal to $|\hat{\mathcal{I}}_2|^{1/k} : |\hat{\mathcal{I}}_1|^{1/k}$.

Experimental Results

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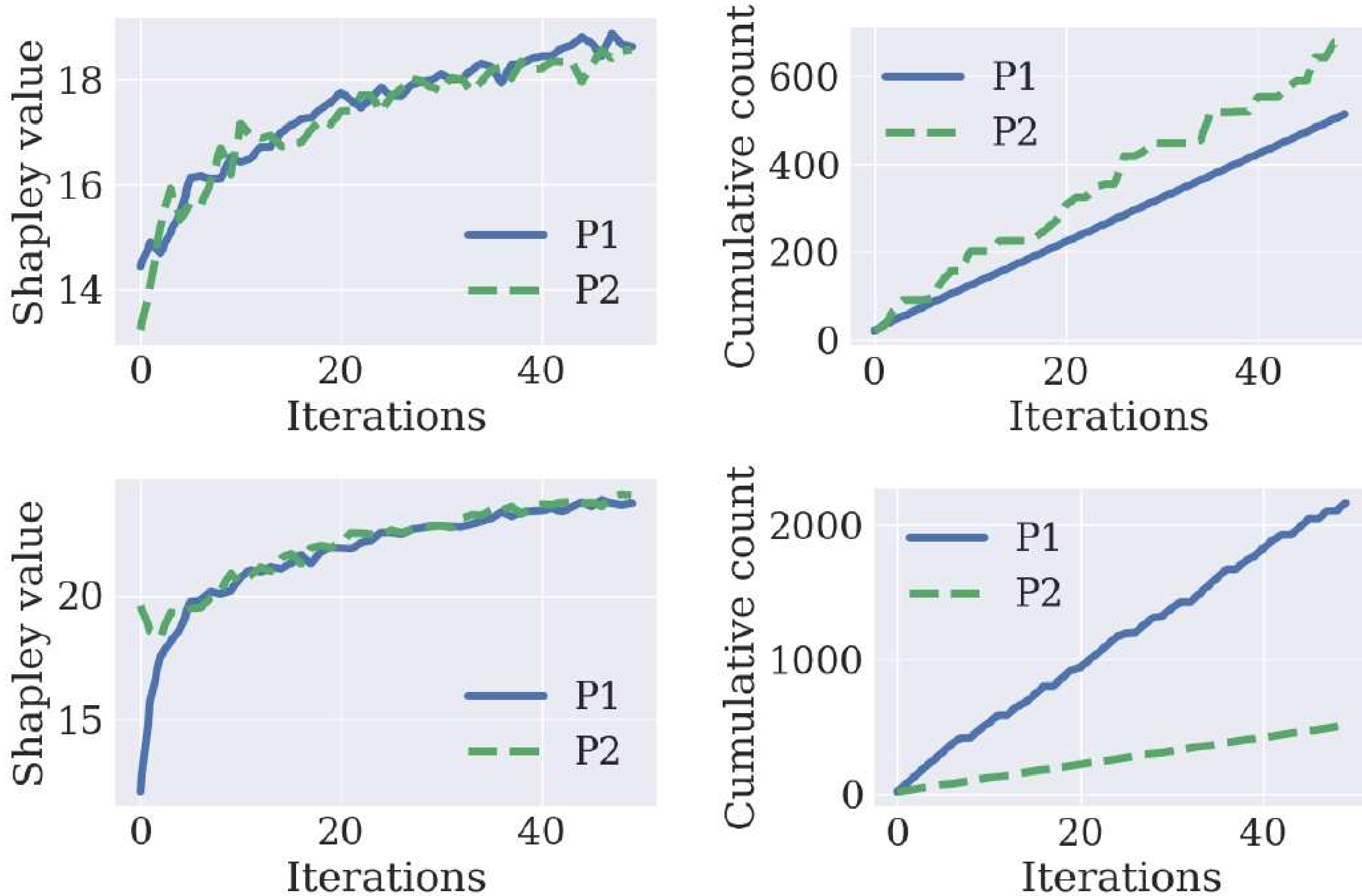

Experimental Results

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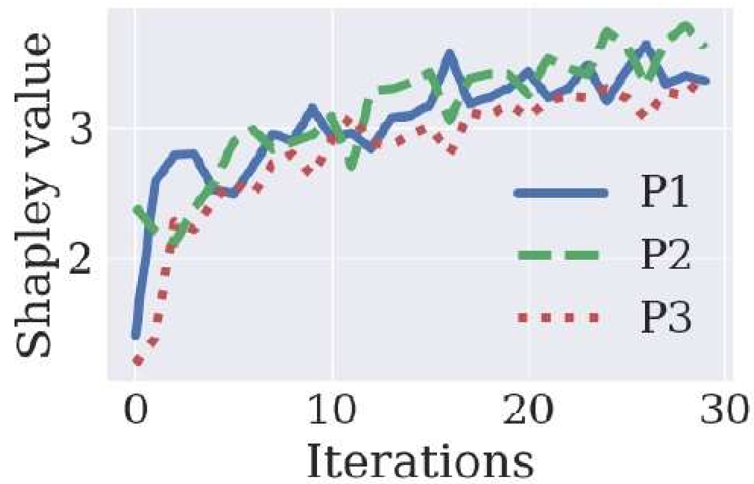

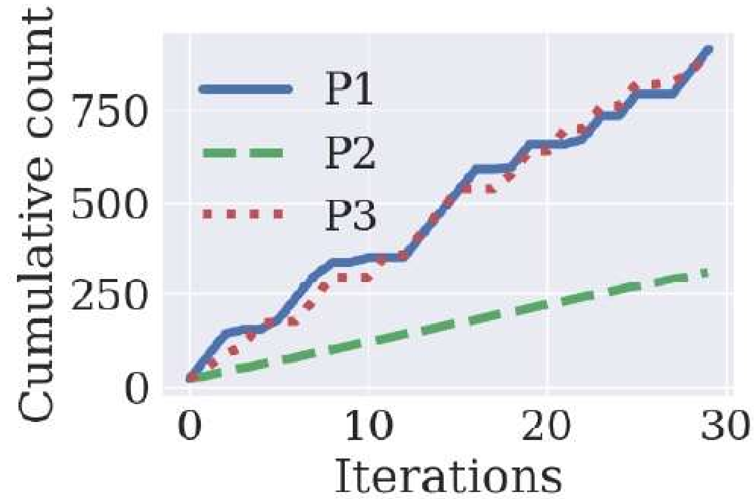

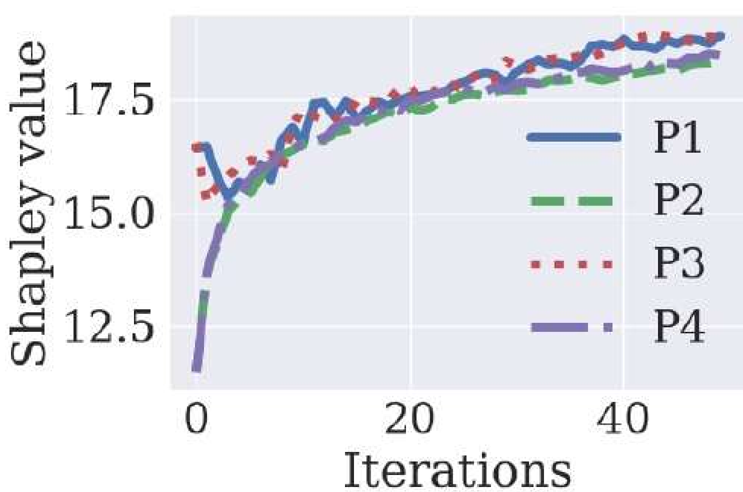

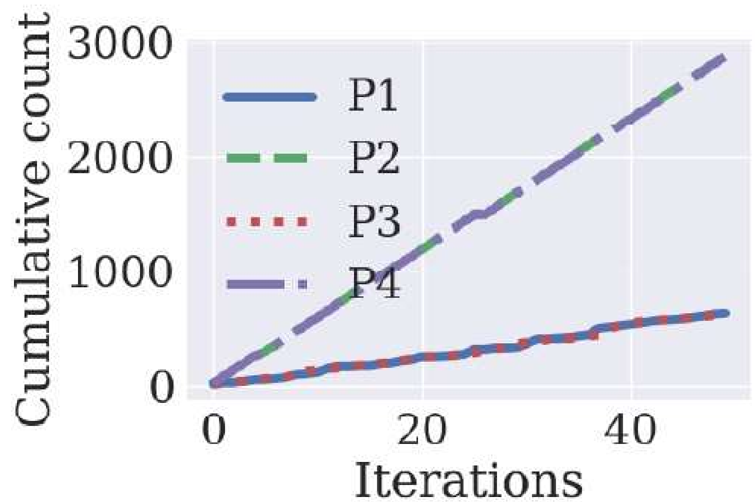

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