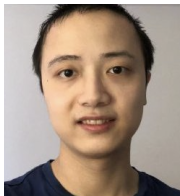# Learning Stochastic Shortest Path with Linear Function Approximation
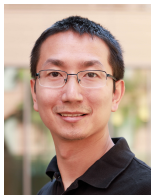
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Stochastic Shortest Path (SSP)

- Online SSP: a type of goal-oriented RL problem
  - Episodic interaction: each episode starts from an initial state and ends when the agent reaches the goal state g
  - Cost: each state-action pair (s, a) incurs a cost $c(s, a)$
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- Beyond tabular SSP: linear function approximation
  - Existing works on tabular SSP (Rosenberg et al. 2020; Cohen et al. 2021; Tarbouriech et al. 2021, ...)
  - Linear mixture SSP: assume that there exists an *unknown* vector $\boldsymbol{\theta}^* \in \mathbb{R}^d$ such that $\mathbb{P}(s'|s, a) = \langle \phi(s'|s, a), \boldsymbol{\theta}^* \rangle$
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This work: efficiently learn linear mixture SSP

Linear Mixture SSP: Algorithmic Design

- Two approaches for SSP in existing literature:
  - By reduction to finite-horizon MDP (Cohen et al. 2021; Chen et al. 2021, ...)
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- LEVIS: a novel optimistic value-iteration algorithm for linear mixture SSP
  - Model estimate updating criteria: coupling features with time
    - Determinant-doubling + time-step-doubling
  - Optimistic planning: contraction via perturbation
    - There is no discount factor in SSP → no contraction for EVI
    - Introduce an auxiliary discount factor by perturbing the transition probability

Linear Mixture SSP: Algorithm

Algorithm 1 LEVIS

1: **for** episode $k = 1, 2, \ldots, K$ **do**
2:    **while** $s_t \neq g$ **do**
3:       Greedily take action a_t, and receive $c(s_t, a_t)$ and s_{t+1}
4:       $\Sigma_t \leftarrow \Sigma_{t-1} + \phi_V(s_t, a_t)\phi_V(s_t, a_t)^\top$
5:       **if** $\det(\Sigma_t)$ or t doubles **then**
6:          Update model estimate $\widehat{\theta}$ and its confidence region
7:          Call DEVI to update estimate of the value functions

Algorithm 2 DEVI

1: **while** $\|V^{(i)} - V^{(i-1)}\|_\infty \geq \epsilon$ **do**
2:    $Q^{(i+1)}(\cdot, \cdot) \leftarrow c_\rho(\cdot, \cdot) + (1 - \rho)\min\langle \theta, \phi_{V^{(i)}}(\cdot, \cdot)\rangle$
3:    $V^{(i+1)}(\cdot) \leftarrow \min_a Q^{(i+1)}(\cdot, a)$

- Determinant-doubling + time-step-doubling
- Perturb the transition probability

Theorem (Regret upper bound)

Under technical assumptions, the proposed algorithm LEVIS achieves a $\widetilde{\mathcal{O}}(dB_\star^{1.5}\sqrt{K/c_{\min}})$ regret, where d is the feature dimension, $B_\star$ is the cost of the optimal policy, $c_{\min} > 0$ is the lower bound of the per-step cost.

Linear Mixture SSP: Theory

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Under technical assumptions, any algorithm for linear mixture SSP incurs at least an expected regret of $\Omega(dB_\star\sqrt{K})$.

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- There is a $\sqrt{B_\star}$-gap between the upper and lower bound. How to do better?

- Design Bernstein-type confidence region to reduce the dependence on $B_\star$
  - Similar technique has been used in online/offline RL (Zhou et al. 2021a; Zhang et al. 2021; Min et al. 2021, ...)

Linear Mixture SSP: Near-optimal Regret

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Theorem (Near-optimal regret bound)

Under technical assumptions, by using a refined Bernstein-type confidence region in algorithm LEVIS, it can achieve $\widetilde{\mathcal{O}}(dB_\star\sqrt{K/c_{\min}})$ regret.

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Theorem (Near-optimal regret bound)

Under technical assumptions, by using a refined Bernstein-type confidence region in algorithm LEVIS, it can achieve $\widetilde{\mathcal{O}}(dB_\star\sqrt{K/c_{\min}})$ regret.

- There is still a remaining gap of $1/\sqrt{c_{\min}}$
- Future work: how to remove the dependence on $c_{\min}$?

THANK YOU!

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