Discrete Probabilistic Inverse Optimal Transport

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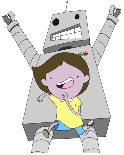

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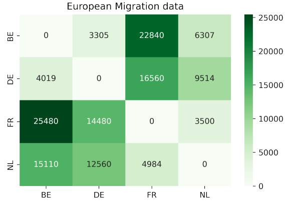
Theory

Motivation

- Inverse OT is the problem of inferring the latent costs C from observed T.
  - It arises naturally in many domains, such as: cost of international migrations.

$T =$ 

$C = ?$ – Existing works.

Is C unique? – No !!

Space of all possible C? – Our work.

- Entropy regularized OT (EOT): a map $\Phi : (C, \mu, \nu) \to T$.
- Inverse OT is defined as $\Phi^{-1} : T \to \{(C, \mu, \nu)\}$ – not unique!!!.
- Goal: We aim to analyze the intrinsic properties of the entire cost space.

Probability Inverse Optimal Transport (PIOT)

Given a noisy observation T of T^*, PIOT is defined as:

$$P(C|T) = \int_{T^* \in (\mathbb{R}^*)^{m \times n}} P(C|T^*)P(T^*|T)dT^*, \quad (1)$$

where $P(C|T^*)$ is obtained through Bayes' Rule:

$$P(C|T^*) = \frac{P(T^*|C)P_0(C)}{P(T^*)}, \quad (2)$$

- Likelihood $P(T^*|C) = 1$ if $C \in \Phi^{-1}(T^*)$, otherwise $P(T^*|C) = 0$;

Support manifold $\Phi^{-1}(T)$ – No noise

Definition. Positive matrices A, B are cross ratio equivalent, $A \overset{c.r.}{\sim} B$, if:

$$r_{ijkl}(A) := \frac{a_{ik}a_{jl}}{a_{il}a_{jk}} = \frac{b_{ik}b_{jl}}{b_{il}b_{jk}} := r_{ijkl}(B). \quad (3)$$

Theorem. Given positive T of dimension $m \times n$, $\Phi^{-1}(T)$ is a hyperplane of dimension $m + n - 1$, consists cost matrices C satisfying $e^{-C} \overset{c.r.}{\sim} T$.

- Supp$[P(C|T)]$ for different priors, incomplete observations, submanifolds.

Support manifold - With noise

Lemma. Given T_1, T_2, hyperplanes $\Phi^{-1}(T_1)$, $\Phi^{-1}(T_2)$ are the same or parallel.

- The distance between IOT of T_1 and T_2 is then well-defined to be the Euclidean distance between $\Phi^{-1}(T_1)$ and $\Phi^{-1}(T_2)$.

Proposition. Given T, assume uniform noise on t_{11} with bounded size a, then

$$\text{supp}[P(C|T)] = \cup_{T' \in \mathbb{B}_a(T)} \Phi^{-1}(T') \text{– a collection of hyperplanes.}$$

Experiments

Approximate inference via MCMC

- Use MCMC method for inferring cost function C
- Advantages:
  - Flexibility on choice of prior $P_0(C)$ or $P_0(K)$
  - Generally inferring a set of possible C instead of a single one
- Procedure: given $K^{(0)} = T$; $K^{(i+1)} = \text{diag}D^r \ K^{(i)} \ \text{diag}D^c$
- Acceptance ratio a depending on the prior:
  - Prior on C: Metropolis (MetroMC): $a = \frac{P(-\ln(K^{(i+1)})|T)}{P(-\ln(K^{(i)}|T)} \overset{(*)}{=} \frac{P_0(C^{i+1},\beta)}{P_0(C^i,\beta)}$
  - Prior on K: Metropolis-Hasting (MHMC): $a = \min\left(1, \frac{P(K')}{P(K)} \frac{Q(K|K'))}{Q(K'|K)}\right)$
- Accept move if $a > u \in [0, 1)$.

Visualizing support of $P(K|T)$

PIOT

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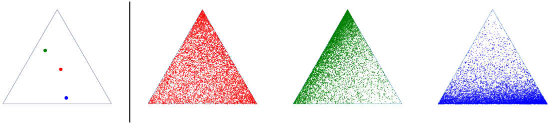
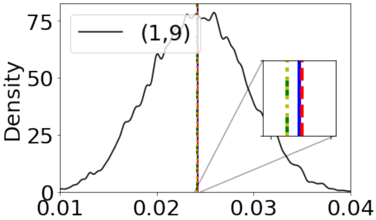
- *Demonstrate effect of cross-ratio equivalence on* $P(K|T)$
- Prior over K is possible and suitable for visualization
- Column-wise Dirichlet prior for $P_0(K)$
  - Visualizing supp$[P(K|T)]$: plotting columns of T & columns of $P(K|T)$



Synthetic data: symmetric cost with noise

- *Accurate cost inference* with relative error $\leq 10^{-3}$;
- $C^g_{ij} = \left|\frac{(i-j)}{n}\right|^p + 10^{-4} * U_{[0,1)}$ (blue)[1]
- Median of the posterior distribution (red)
- Compare with Ma et al. (arXiv:2002.09650) (yellow & green)



[1] The inset in the figure has a range of $C^g \pm 10^{-4}$.

Real-world data: European migration flow (1)

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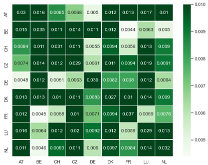
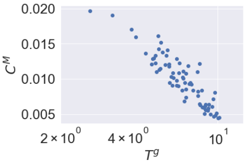
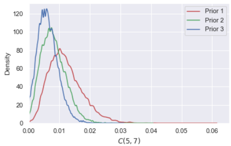
- Ground truth cost not available
- Inferred cost inversely proportional to log of migration flow
- Consider 3 priors, stronger prior gives sharper posterior
- *PIOT with soft constrain is less biased*

Ground truth T^g



Inferred cost C





Real-world data: European migration flow (2)

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- Evaluate predictability of PIOT on 2 cases: T with (1) noise (2) missing value
  - Procedure: T –[PIOT]→ $P(C|T)$ –[EOT]→ $P(T')$; $T^p = \overline{P(T')}$
  - NOISE: $T_{ij} = T^g_{ij} + \epsilon_{ij}$; MISSING VALUE: $T_{ij} \in [100, 25000)$
- *Accurate predictions on noisy/missing elements* → *PIOT infers cost well*

	Noise			Missing value		
	(3,6)	(4,5)	(5,7)	(3,6)	(4,5)	(5,7)
T^g	452	6635	16560	452	6635	16560
T^p	454	6593	16515	3699	7462	12349
$\|\Delta\|$	.45%	.87%	.25%	26%	6.6%	34%

Table: Comparison between the predicted coupling, T^p, with the ground truth coupling, T^g. $\Delta = (T^p - T^g)/\overline{T}^g$. Corresponding element is corrupted by a 4% Gaussian noise on the left column, and missing on the right column.

Conclusions

Conclusions

In this work we

- formulate PIOT and identify the support manifold for latent costs
- characterize the geometry of the support manifold
- analyze the support manifold for noisy and incomplete observations
- develop MCMC algorithms for posterior inference
- demonstrate PIOT through simulations on general examples

Thank you!

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