Jul 17–23, 2022

Uncertainty Modeling in Generative Compressed Sensing

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Acknowledgements: Mengchu Xu, Prof. Xiaojun Mao, and Prof. Jian Wang*

NSFC grants 12001109, 92046021, 61971146

Y. Zhang, M. Xu, X. Mao, and J. Wang, "Uncertainty Modeling in Generative Compressed Sensing," *Proceedings of the 39th International Conference on Machine Learning*, 2022.

Sparsity-based Compressed Sensing

- Inverse problems

Goal: recover signal $\mathbf{x} \in \mathbb{R}^n$ from compressed linear measurements

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n} \qquad \mathbf{A} \in \mathbb{R}^{m \times n} \ (m \ll n)$$

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Limitation: $\mathbf{x}$ is not strictly sparse ➤ inaccurate recovery results

Generative Compressed Sensing

- Compressed sensing with generative models (CSGM)

Key idea: learn more accurate priors in a data-driven way

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S1. Pre-train a generator (GAN) / decoder (VAE) using training signals $\mathbf{X}_{\mathrm{tr}} = [\mathbf{x}_1, \ldots, \mathbf{x}_N]$

$$\mathbf{g}(\mathbf{z}; \hat{\boldsymbol{\theta}}) : \mathbb{R}^k \mapsto \mathbb{R}^n$$

latent variables $\mathbf{z} \in \mathbb{R}^k$ $(k \ll n)$
with prior $\mathcal{N}(\mathbf{z}; \mathbf{0}, \mathbf{I}_k)$

- output signals that resemble the training ones

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S2. Optimize latent variable to minimize the loss

$$\hat{\mathbf{z}} = \arg\min_{\mathbf{z}} \|\mathbf{A}\mathbf{g}(\mathbf{z}; \hat{\boldsymbol{\theta}}) - \mathbf{y}\|_2^2 + \lambda_{\mathbf{z}} \|\mathbf{z}\|_2^2$$

relative weight $\lambda_{\mathbf{z}}$

and then reconstruct

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✓ Bounded reconstruction error for $\mathbf{x}$ inside the range $\mathcal{R}(\mathbf{g}(\mathbf{z}; \hat{\boldsymbol{\theta}})) := \{\mathbf{g}(\mathbf{z}; \hat{\boldsymbol{\theta}}) | \mathbf{z} \in \mathbb{R}^k\}$

$$\gamma \|\hat{\mathbf{x}} - \mathbf{x}\|_2 - \epsilon \leq \|\mathbf{A}\mathbf{g}(\hat{\mathbf{z}}; \hat{\boldsymbol{\theta}}) - \mathbf{y}\|_2$$

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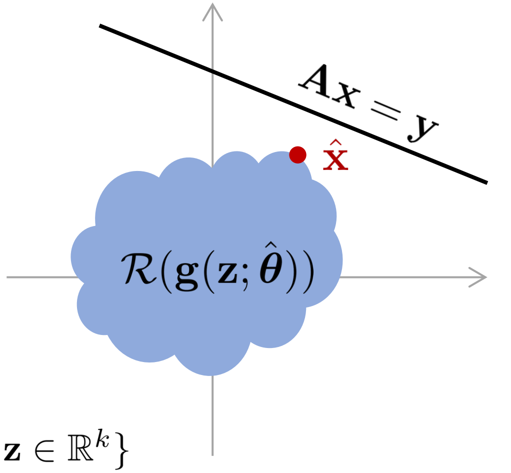

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❖ Inferior performance when $\mathbf{x} \notin \mathcal{R}(\mathbf{g}(\mathbf{z}; \hat{\boldsymbol{\theta}}))$

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Modeling Uncertainties in CSGM

Key idea: deterministic $\hat{\boldsymbol{\theta}}$ leads to fixed range $\mathcal{R}(\mathbf{g}(\mathbf{z};\hat{\boldsymbol{\theta}}))$ ➢ model uncertainties in $\boldsymbol{\theta}$

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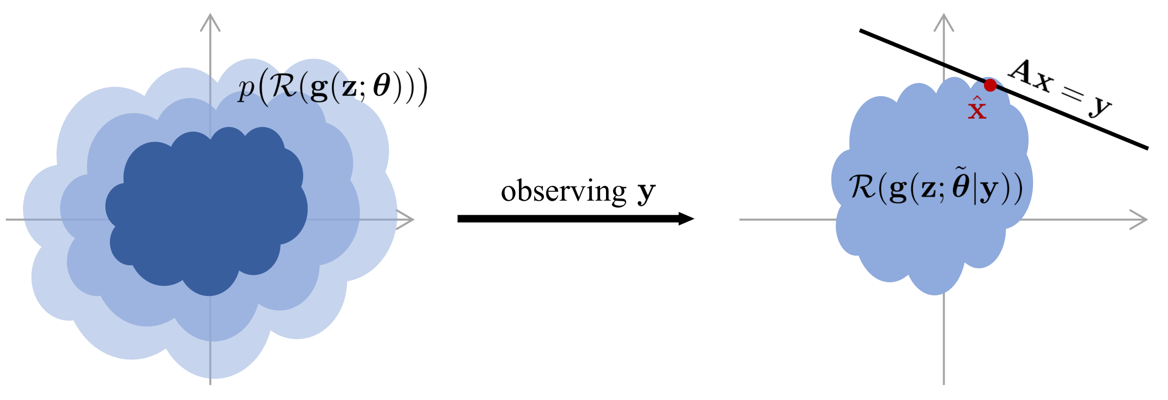

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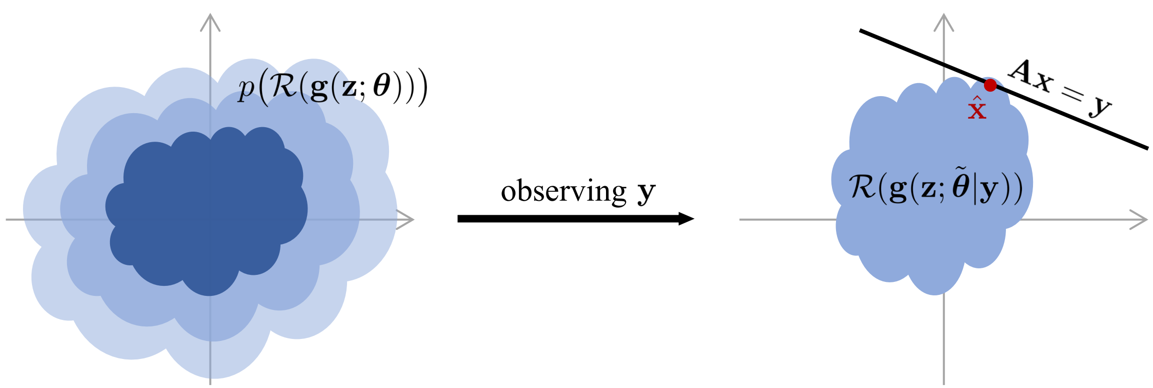
  - Prediction $\mathbb{E}[\mathbf{x}|\mathbf{y};\mathbf{X}_{\text{tr}}] = \mathbb{E}_{p(\mathbf{z},\boldsymbol{\theta}|\mathbf{y};\mathbf{X}_{\text{tr}})}[\mathbf{g}(\mathbf{z};\boldsymbol{\theta})]$



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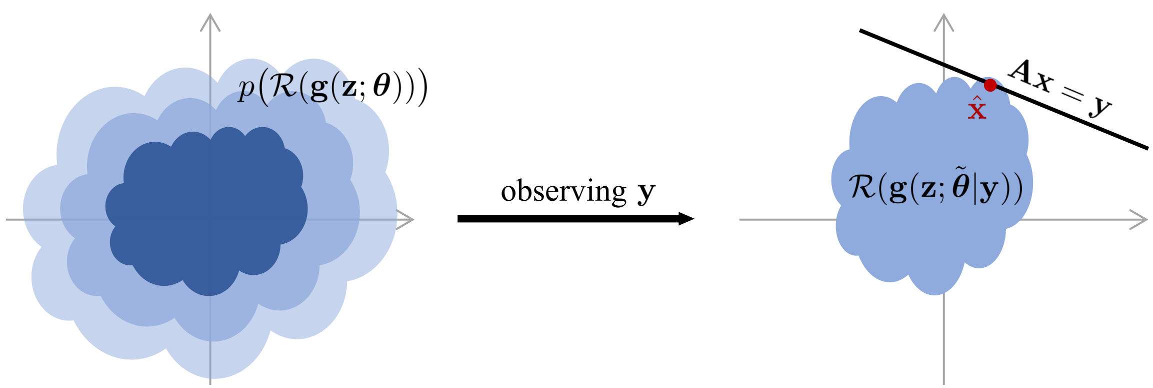
- Solve for $p(\mathbf{z},\boldsymbol{\theta}|\mathbf{y};\mathbf{X}_{\text{tr}})$ via alternating optimization
  - maximum a posteriori (MAP) for high-dimensional $\boldsymbol{\theta}$
  - variational inference (VI) for low-dimensional $\mathbf{z}$



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Theoretical Justification of CS-BGM

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- Necessary condition for generator range
  - ✓ (Bora et al'17) If $\mathbf{x} \in \mathcal{R}(\mathbf{g}(\mathbf{z}; \hat{\boldsymbol{\theta}}))$, small reconstruction error can be achieved

$$\text{S-REC: } \gamma \|\hat{\mathbf{x}} - \mathbf{x}\|_2 - \epsilon \leq \|\mathbf{A}\mathbf{g}(\hat{\mathbf{z}}; \hat{\boldsymbol{\theta}}) - \mathbf{y}\|_2$$

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$$\text{S-RIP: } \sqrt{1-\delta}\|\hat{\mathbf{x}} - \mathbf{x}\|_2 + \epsilon \leq \|\mathbf{A}\mathbf{g}(\hat{\mathbf{z}}; \hat{\boldsymbol{\theta}}) - \mathbf{y}\|_2 \leq \sqrt{1+\delta}\|\hat{\mathbf{x}} - \mathbf{x}\|_2 + \epsilon$$

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	CSGM	CS-BGM
Measurement error $\frac{1}{n}\|\mathbf{A}\mathbf{g}(\hat{\mathbf{z}};\hat{\boldsymbol{\theta}}) - \mathbf{y}\|_2^2$:	1.073	0.047
Reconstruction error $\frac{1}{n}\|\mathbf{g}(\hat{\mathbf{z}};\hat{\boldsymbol{\theta}}) - \mathbf{x}\|_2^2$:	0.0137	0.0044

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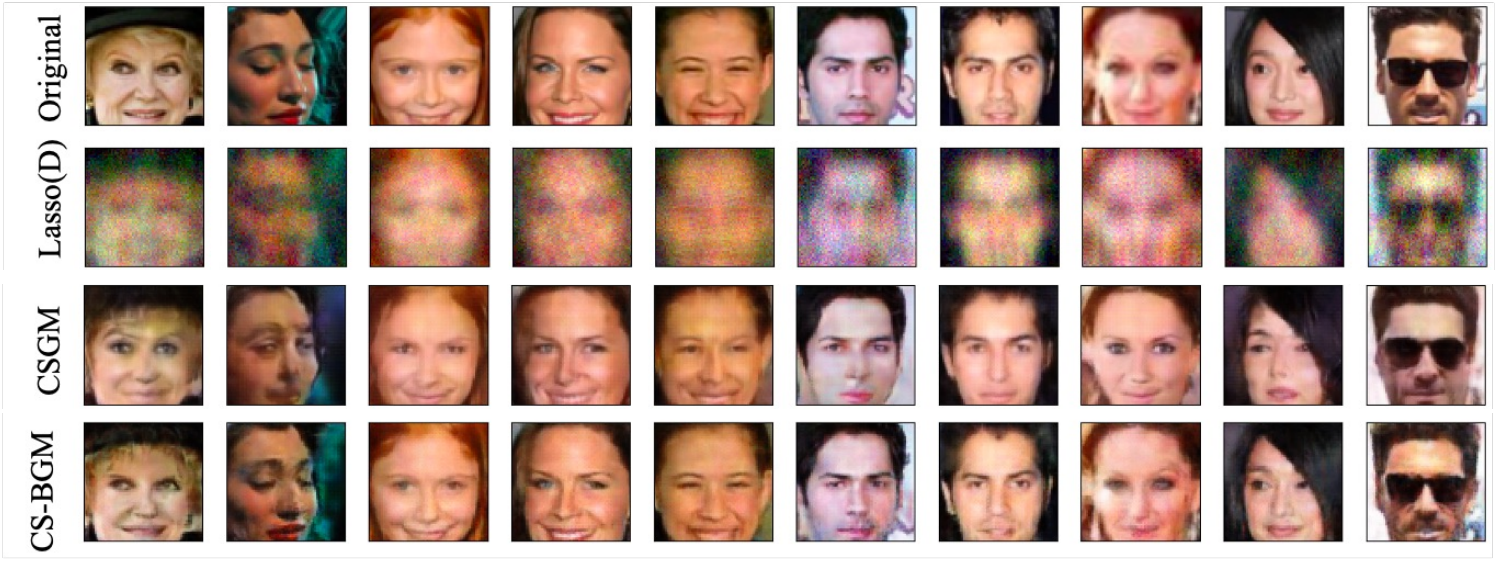
Experiments



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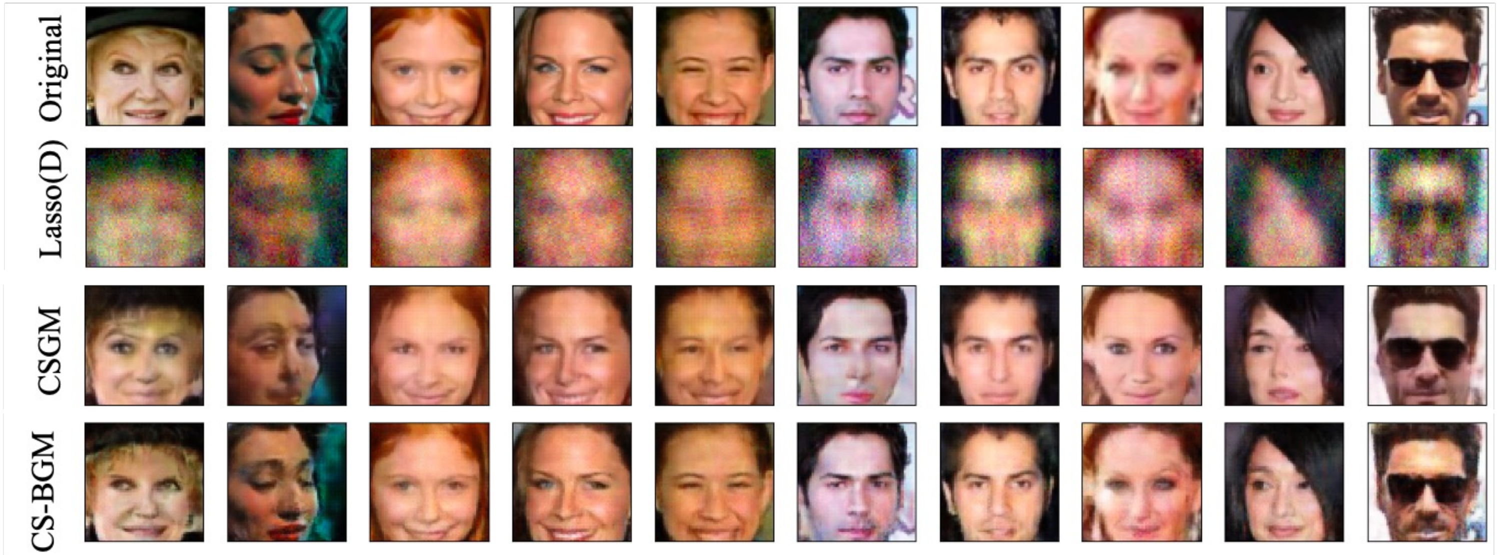

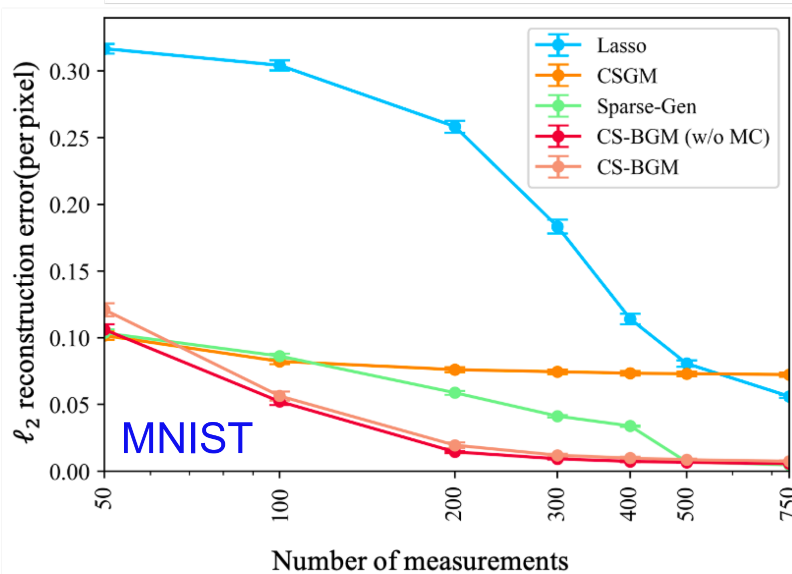

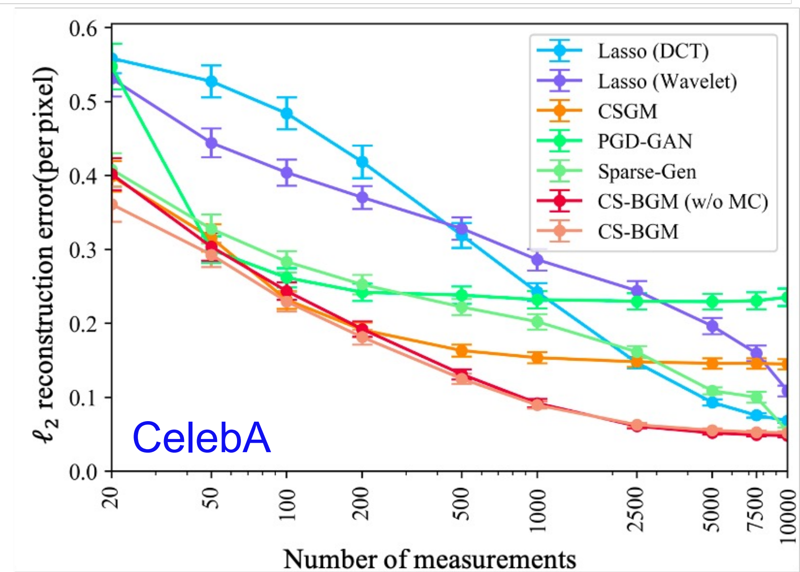

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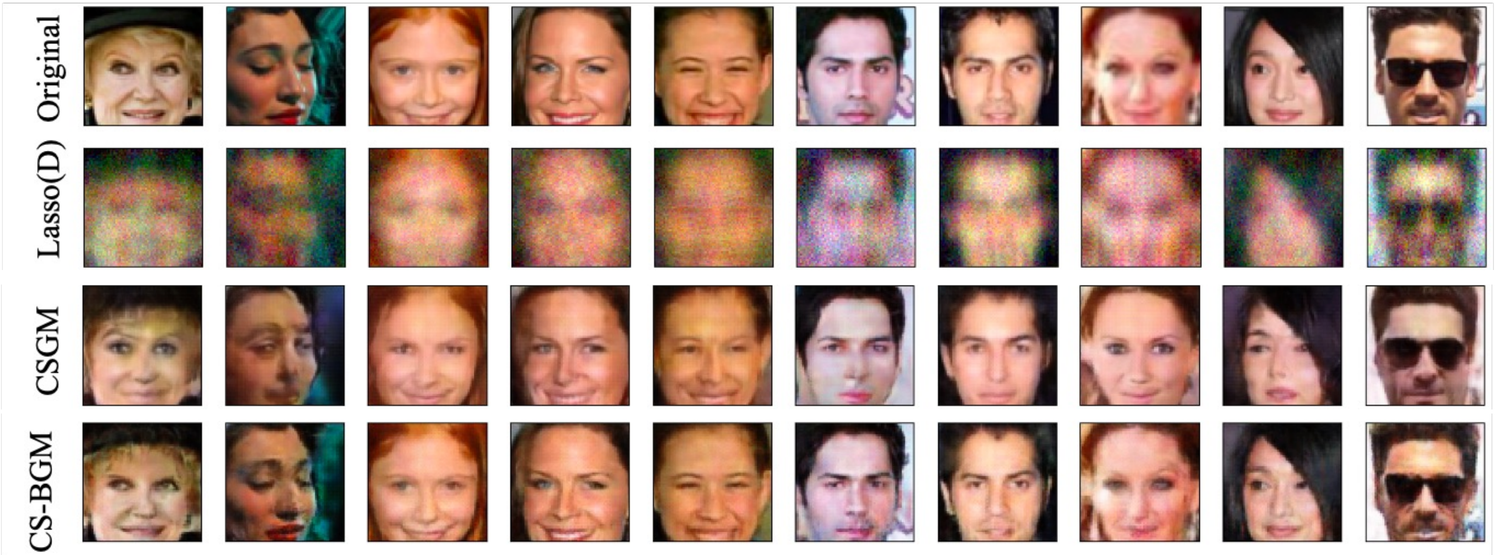

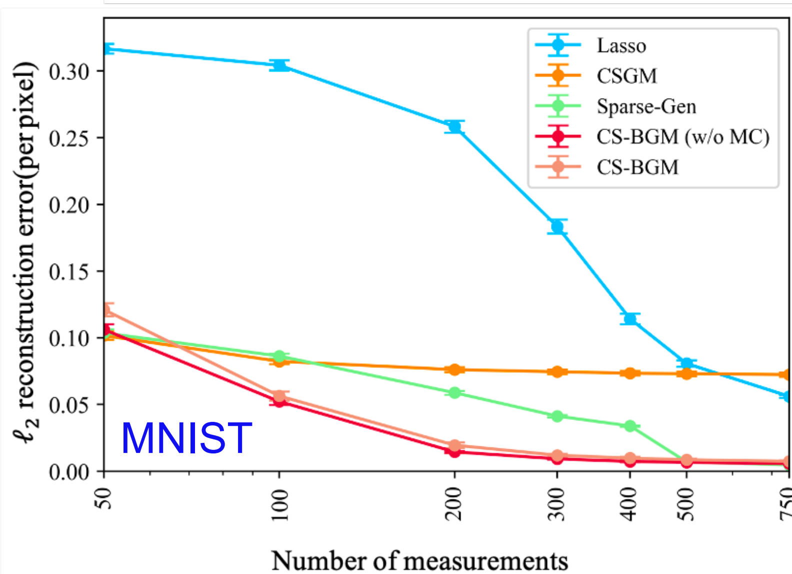

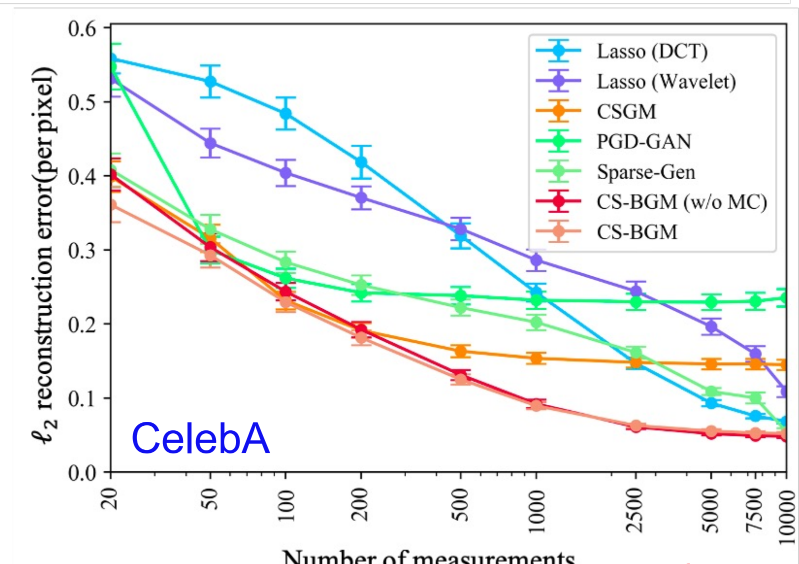

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