Faster Algorithms for Learning Convex Functions

Ali Siahkamari, Durmus Alp Emre Acar, Christopher Liao, Kelly Geyer, Venkatesh Saligrama, Brian Kulis

ICML 2022, Baltimore, USA

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Problem setup

- Input:

$$(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n) \quad | \quad \mathbf{x}_i \in \mathbb{R}^d, y_i \in \mathbb{R} \,.$$

- Model:

$$\mathcal{F} \triangleq \{f : \mathbb{R}^d \rightarrow \mathbb{R} \mid f \text{ is convex}\} \,.$$

- Estimation rule:

$$\hat{f} \triangleq \arg\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} (y_i - f(\mathbf{x}_i))^2 + \lambda \|f\|,$$

where, $\|f\| \triangleq \sum_{l=1}^{d} \sup_{\mathbf{x}} \sup_{v \in \partial_{x_l} f(\mathbf{x})} |v|$

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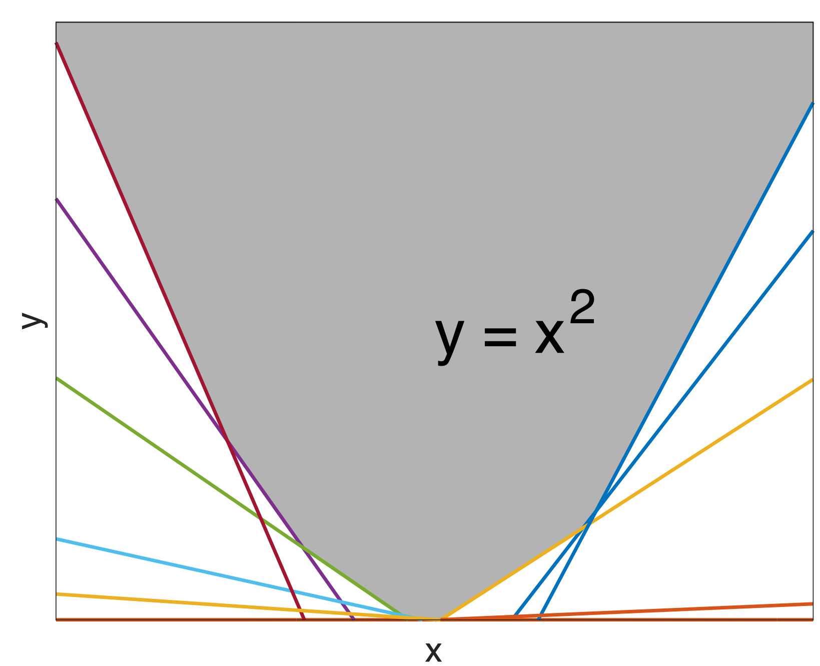
Convex formulation

Using [Boyd & Vandenberghe (2004)], the estimation rule for $\hat{f}$ is equivalent to solving:

$$\min_{\hat{y}_i, \mathbf{a_i}} \sum_{i=1}^{n} (\hat{y}_i - y_i)^2 + \lambda \sum_{l=1}^{d} \max_{i=1}^{n} |a_{i,l}|$$

$$s.t: \hat{y}_i - \hat{y}_j \geq \langle \mathbf{a}_j, \mathbf{x}_i - \mathbf{x}_j \rangle, \quad i, j \in [n].$$



And then constructing $\hat{f}$ as:

$$\hat{f}(\mathbf{x}) \triangleq \max_{i=1}^{n} \langle \mathbf{a}_i, \mathbf{x} - \mathbf{x}_i \rangle + \hat{y}_i$$

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Convergence

Theorem 1: [from He & Yuan (2012)] Consider the separable convex optimization problem,

$$\min_{\mathbf{b}^1 \in \mathcal{S}_1, \mathbf{b}^2 \in \mathcal{S}_2} \left[\psi(\mathbf{b}^1, \mathbf{b}^2) = \psi_1(\mathbf{b}^1) + \psi_2(\mathbf{b}^2)\right] \qquad s.t: \mathbf{A}\mathbf{b}^1 + \mathbf{B}\mathbf{b}^2 + \mathbf{b} = \mathbf{0},$$

Let $\mathbf{b}_t^1$ and $\mathbf{b}_t^2$ be solutions at iteration t of a two block ADMM procedure with learning rate ρ.

Denote: $(\tilde{\mathbf{b}}_T^1, \tilde{\mathbf{b}}_T^2) = \left(\frac{1}{T}\sum_{t=1}^{T} \mathbf{b}_t^1, \frac{1}{T}\sum_{t=1}^{T} \mathbf{b}_t^2\right)$ and $(\mathbf{b}_1^*, \mathbf{b}_2^*)$ the optimal solution. For all κ we have:

$$\psi(\tilde{\mathbf{b}}_T^1, \tilde{\mathbf{b}}_T^2) - \psi(\mathbf{b}_*^1, \mathbf{b}_*^2) - \kappa^T(\mathbf{A}\tilde{\mathbf{b}}_T^1 + \mathbf{B}\tilde{\mathbf{b}}_T^2 + \mathbf{b}) \leq \frac{\mathbf{1}}{\mathbf{T}}\left(\frac{\rho}{\mathbf{2}}\|\mathbf{B}\mathbf{b}_*^{\mathbf{2}}\|^{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{2}\rho}\|\kappa\|^{\mathbf{2}}\right),$$

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Convergence

Theorem 2: Let $\{\hat{y}_i^t, \mathbf{a}_i^t\}_{i=1}^n$ be the output of convex regression ADMM at t^{th} iteration,

Set $\tilde{y}_i \triangleq \frac{1}{T}\sum_{t=1}^{T} \hat{y}_i^t$, $\tilde{\mathbf{a}}_i \triangleq \frac{1}{T}\sum_{t=1}^{T} \mathbf{a}_i^t$ and $\rho = \frac{\sqrt{d}\lambda^2}{n}$

Denote. $\tilde{f}_T(\mathbf{x}) \triangleq \max_i \langle \tilde{\mathbf{a}}_i, \mathbf{x} - \mathbf{x}_i \rangle + \tilde{y}_i$.

Assume $\max_{i,l} |x_{i,l}| \leq 1$ and $Var(\{y_i\}_{i=1}^n) \leq 1$. For $\lambda \geq \frac{3}{\sqrt{2nd}}$ and $T \geq n\sqrt{d}$ we have:

$$\frac{1}{n}\sum_{i=1}^{n} (\tilde{f}_T(\mathbf{x}_i) - y_i)^2 + \lambda \|\tilde{f}_T\| \leq \min_{f \in \mathcal{F}} \left(\frac{1}{n}\sum_{i=1}^{n} \big(f(\mathbf{x}_i) - y_i\big)^2 + \lambda \|f\| \right) + \frac{6n\sqrt{d}}{T+1}$$

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Computational complexity

SOTA computational complexity for Lipschitz convex regression:

$$O\left(\frac{n^5 d^2}{\epsilon}\right).$$

Out result:

$$O\left(\frac{n^3 d^{1.5} + n^2 d^{2.5} + n d^3}{\epsilon}\right).$$

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Experiments

Table 1. Comparison of Convex Regression run time against baseline on Synthetic Data

		Seconds	
n	d	Baseline (Siahkamari et al., 2020)	This Paper
1000	2	32.3	3.63
1000	4	30.8	3.75
1000	8	54.5	4.03
1000	16	112.3	4.12
1000	32	189.3	4.37

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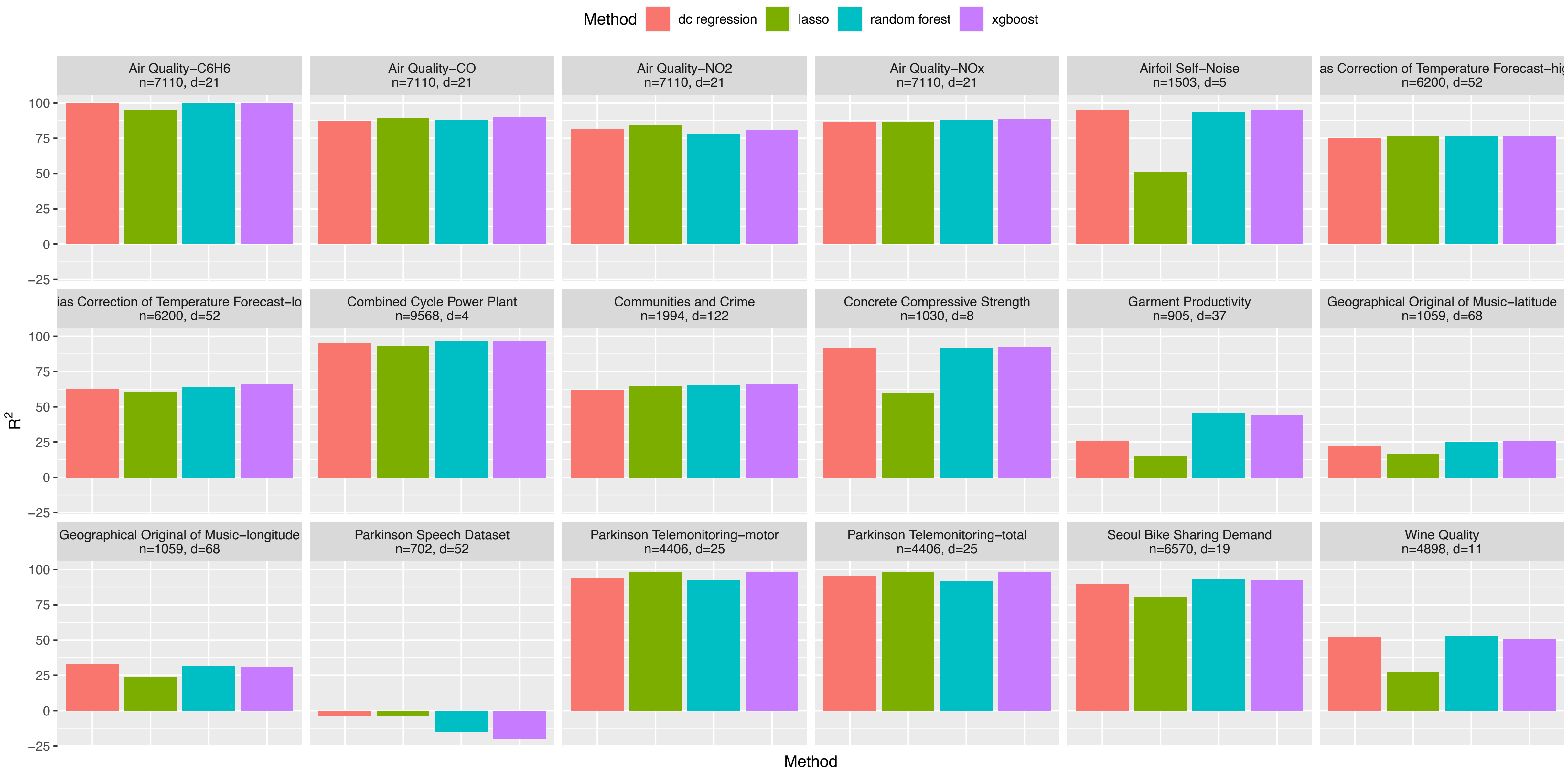
Experiments with DC regression - R^2



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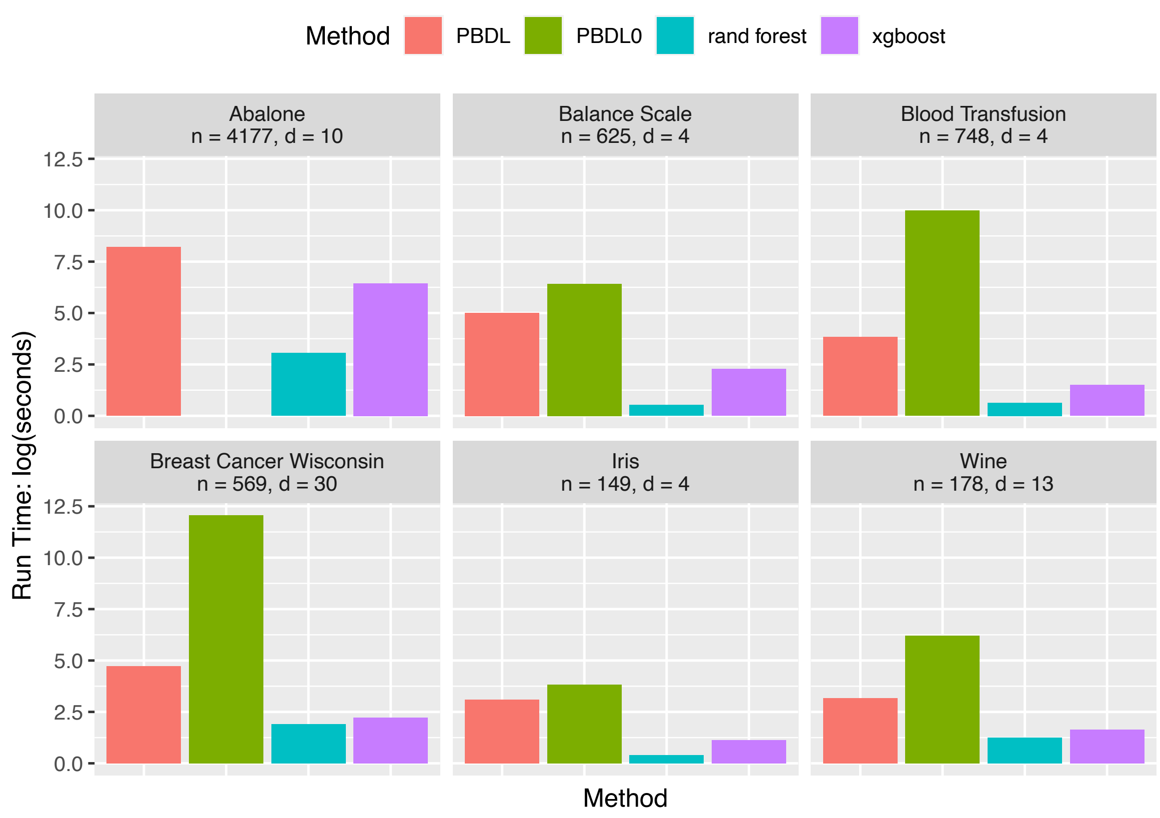
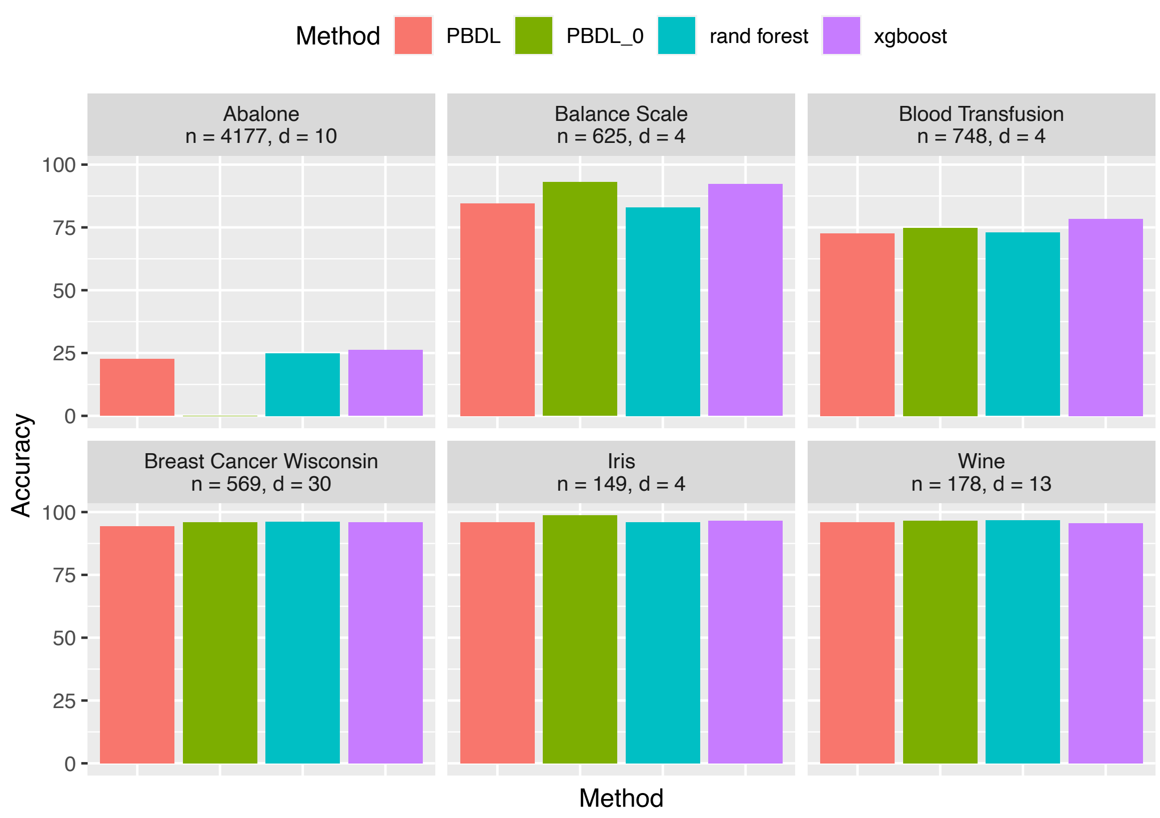
Experiments with Learning Bregman Divergences





Thank you

- Check out our paper: Faster Algorithms for Learning Convex function