Robust SDE-Based Variational Formulations for Solving Linear PDEs via Deep Learning

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(equal contribution)

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Scope: Linear Kolmogorov PDEs

- We want to solve PDEs (*partial differential equations*) of the form

$$\left(\partial_t + \frac{1}{2}(\sigma\sigma^\top) : \nabla^2 + b \cdot \nabla\right) V(x,t) = 0, \qquad V(x,T) = g(x), \qquad (x,t) \in \mathbb{R}^d \times [0,T].$$

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- **Idea:** minimize *variational formulations* using neural networks $u_\theta \in \mathcal{U}$ with parameters θ, i.e. consider losses

$$\mathcal{L} : \mathcal{U} \to \mathbb{R}_{\geq 0},$$

which shall be minimal iff $u \in \mathcal{U}$ fulfills the PDE.

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where X is the solution to the SDE (*stochastic differential equation*)

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- As the stochastic integral S_u has vanishing expectation, this motivates the two losses

$$\mathcal{L}_{\mathrm{FK}}(u) := \mathbb{E}\left[\Delta_u^2\right] \qquad \text{and} \qquad \mathcal{L}_{\mathrm{BSDE}}(u) := \mathbb{E}\left[(\Delta_u - S_u)^2\right],$$

  where $(\xi,\tau) \sim \mathrm{Unif}(\mathbb{R}^d \times [0,T])$.

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Proposition (Variance of Losses)

$$\mathbb{V}\left[\mathcal{L}_{\mathrm{FK}}^{(K)}(u_\theta)\right] = \frac{1}{K}\,\mathbb{V}\left[S_V^2\right] \qquad \textit{and} \qquad \mathbb{V}\left[\mathcal{L}_{\mathrm{BSDE}}^{(K)}(u_\theta)\right] = 0.$$

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Proposition (Variance of Gradients)

$$\mathbb{V}\left[\nabla_\theta \mathcal{L}_{\mathrm{FK}}^{(K)}(u_\theta)\right] = \frac{4}{K}\mathbb{V}\left[S_V \nabla_\theta u_\theta(\xi,\tau)\right] \qquad \textit{and} \qquad \mathbb{V}\left[\nabla_\theta \mathcal{L}_{\mathrm{BSDE}}^{(K)}(u_\theta)\right] = 0.$$

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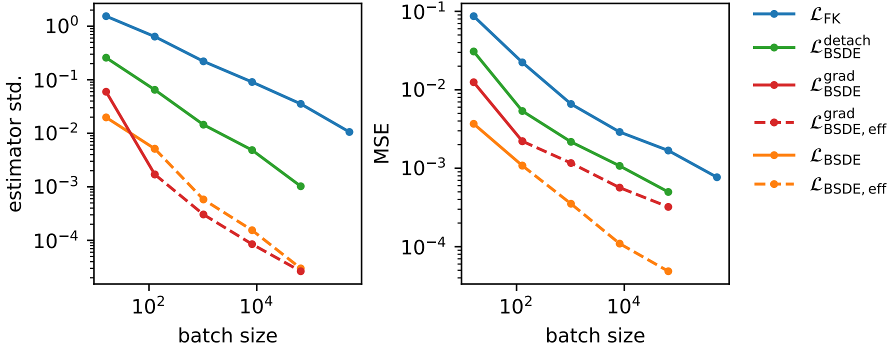
- We improve state-of-the-art performance and analyze trade-offs between accuracy and complexity.



Thank you for your attention!

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Source code: `https://github.com/juliusberner/robust_kolmogorov`