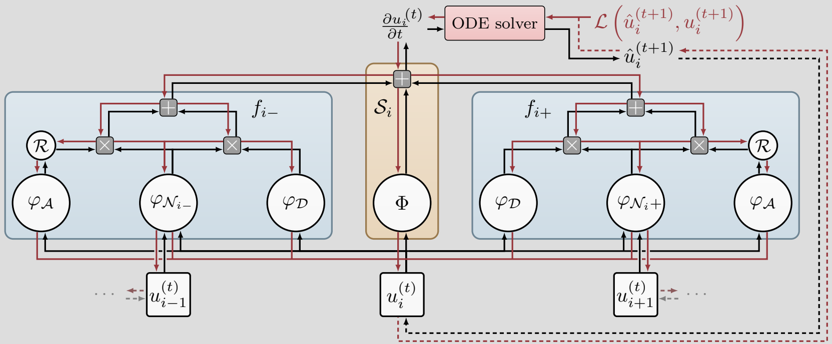
EBERHARD KARLS UNIVERSITÄT TÜBINGEN
Faculty of Natural Sciences
Computer Science · Wilhelm-Schickard-Institut · Neuro-Cognitive Modeling

Institute for Modelling Hydraulic and Environmental Systems
University of Stuttgart Germany
Department of Stochastic Simulation and Safety Research for Hydrosystems



$$\frac{\partial u_i}{\partial t} = \underbrace{\underbrace{- \overbrace{v(u_i)}^{\mathcal{R}(\varphi_\mathcal{A})} \overbrace{\frac{\partial u_i}{\partial x}}^{\varphi_{\mathcal{N}_{i-}} \text{ or } \varphi_{\mathcal{N}_{i+}}} + \overbrace{D(u_i)}^{\varphi_\mathcal{D}} \overbrace{\frac{\partial^2 u_i}{\partial x^2}}^{\varphi_{\mathcal{N}_{i-}} + \varphi_{\mathcal{N}_{i+}}}}_{\mathcal{F}_i = f_{i-} + f_{i+}} + \overbrace{q(u_i)}^{\Phi}}_{\mathcal{S}_i}$$

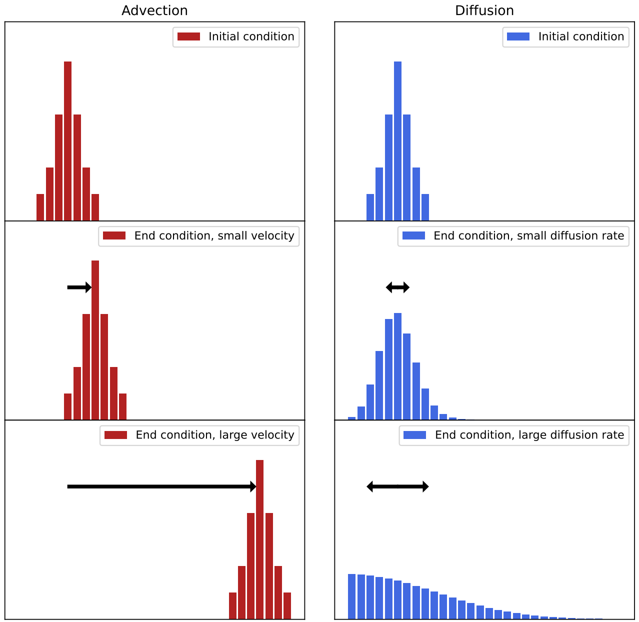
Composing Partial Differential Equations with Physics-Aware Neural Networks

Matthias Karlbauer*, Timothy Praditia*, Sebastian Otte, Sergey Oladyshkin, Wolfgang Nowak, Martin V. Butz

* Equal contribution

SimTech Cluster of Excellence
impRS-IS

Advection-Diffusion Processes



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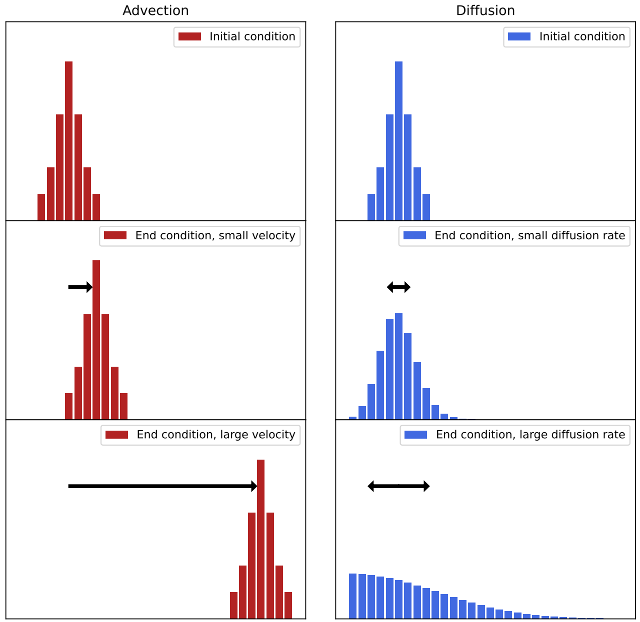
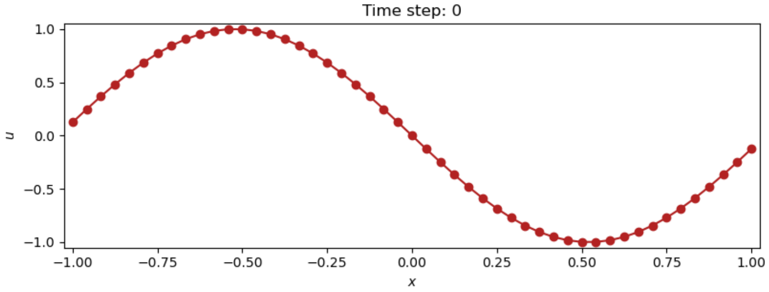
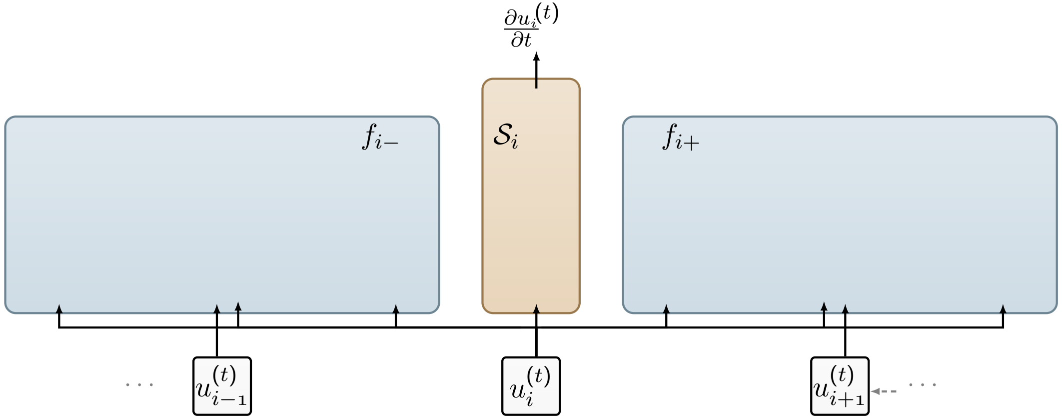


$$\frac{\partial u}{\partial t} = \underbrace{-v(u)\frac{\partial u}{\partial x}}_{\text{Advection}} + \underbrace{D(u)\frac{\partial^2 u}{\partial x^2}}_{\text{Diffusion}}$$



FInite volume Neural Network (FINN)



$$\frac{\partial u_i}{\partial t} = - v(u_i) \frac{\partial u_i}{\partial x} + D(u_i) \frac{\partial^2 u_i}{\partial x^2} + q(u_i)$$

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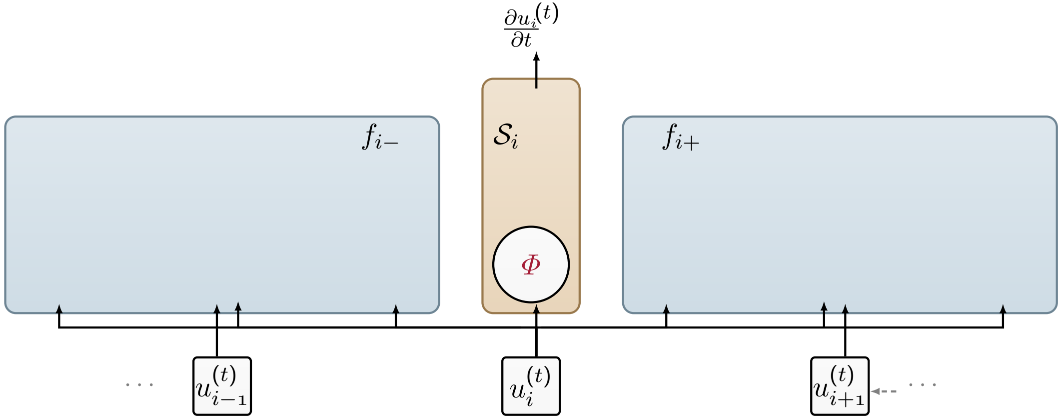

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$\Phi^*(u_i)$

*Either a learnable parameter, a neural network, or (if known) the physical term.

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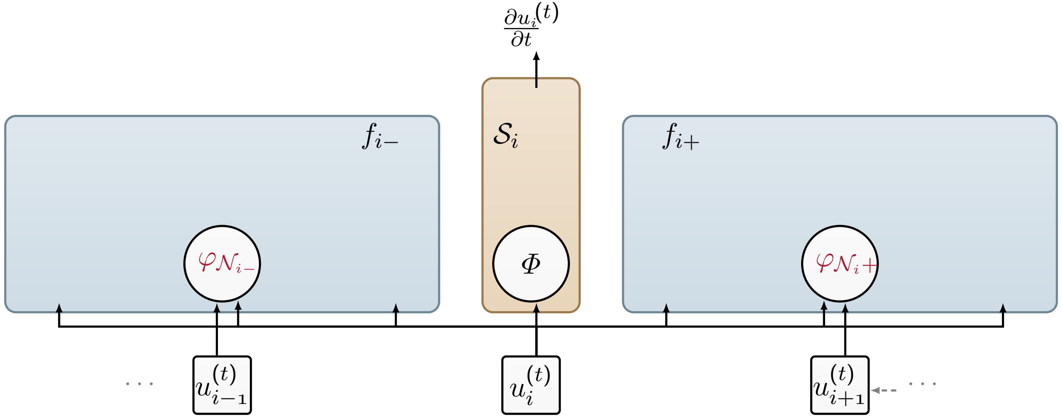

$$\frac{\partial u_i}{\partial t} = -\, v(u_i)\, \overbrace{\frac{\partial u_i}{\partial x}}^{\varphi\mathcal{N}_{i-} \text{ or } \varphi\mathcal{N}_{i+}} + D(u_i)\, \overbrace{\frac{\partial^2 u_i}{\partial x^2}}^{\varphi\mathcal{N}_{i-} + \varphi\mathcal{N}_{i+}} + \overbrace{q(u_i)}^{\Phi}$$

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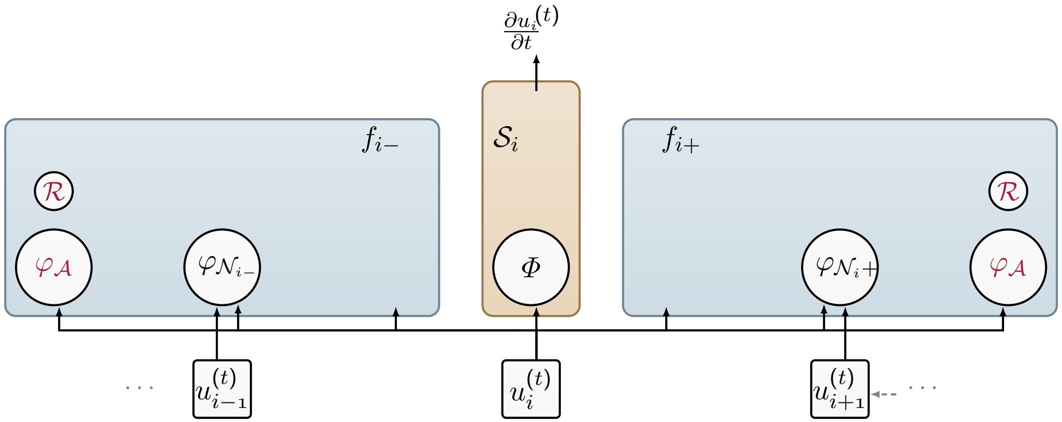

$$\frac{\partial u_i}{\partial t} = -\overbrace{v(u_i)}^{\mathcal{R}(\varphi_{\mathcal{A}})}\overbrace{\frac{\partial u_i}{\partial x}}^{\varphi_{\mathcal{N}_{i-}} \text{ or } \varphi_{\mathcal{N}_{i+}}} + \overbrace{D(u_i)}\overbrace{\frac{\partial^2 u_i}{\partial x^2}}^{\varphi_{\mathcal{N}_{i-}} + \varphi_{\mathcal{N}_{i+}}} + \overbrace{q(u_i)}^{\Phi}$$

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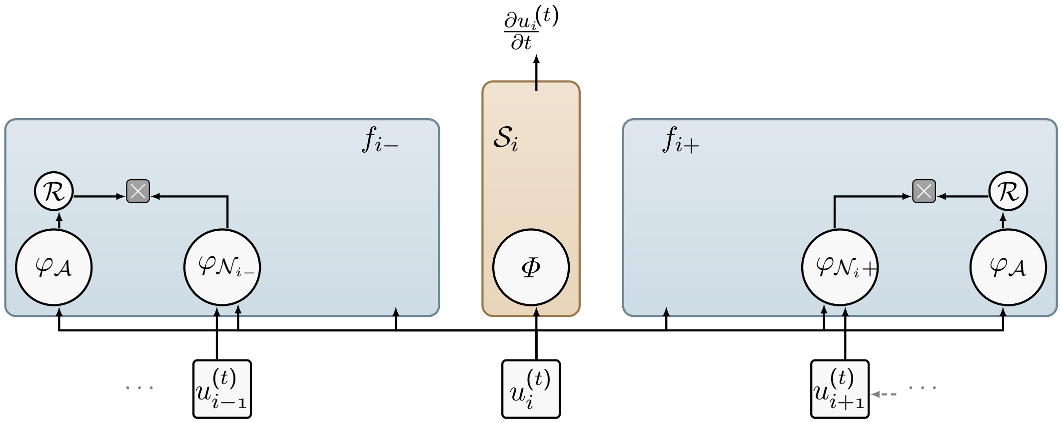

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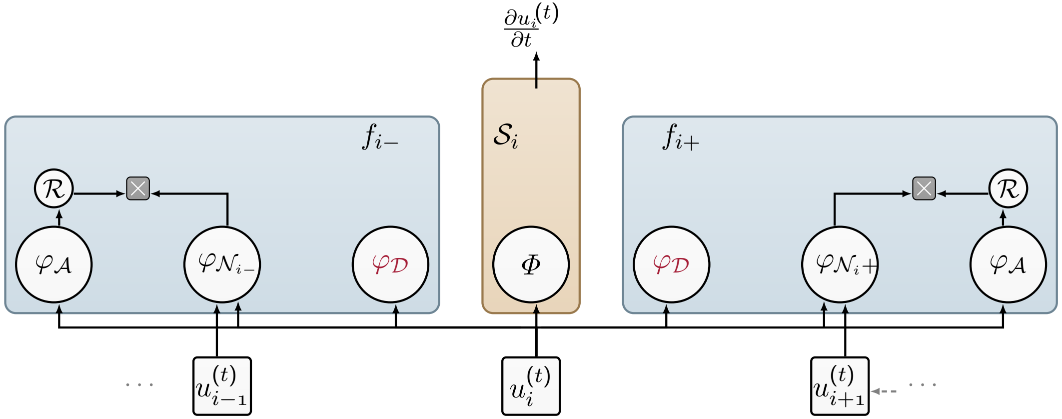

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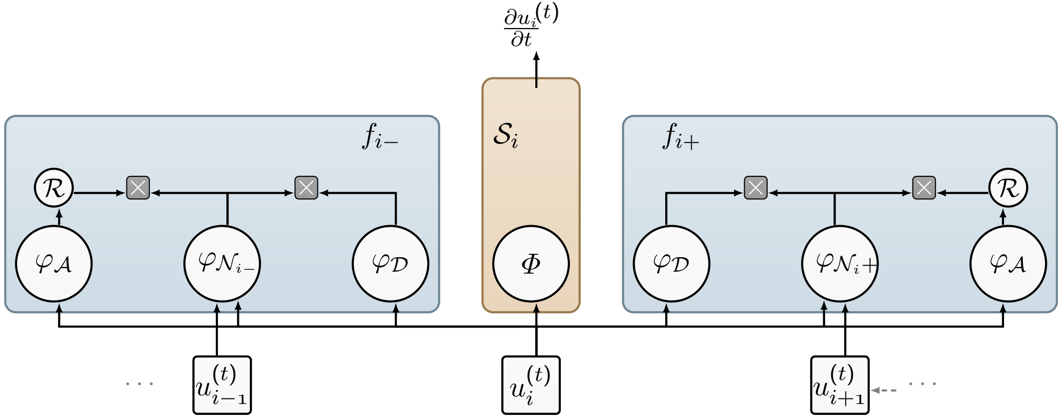

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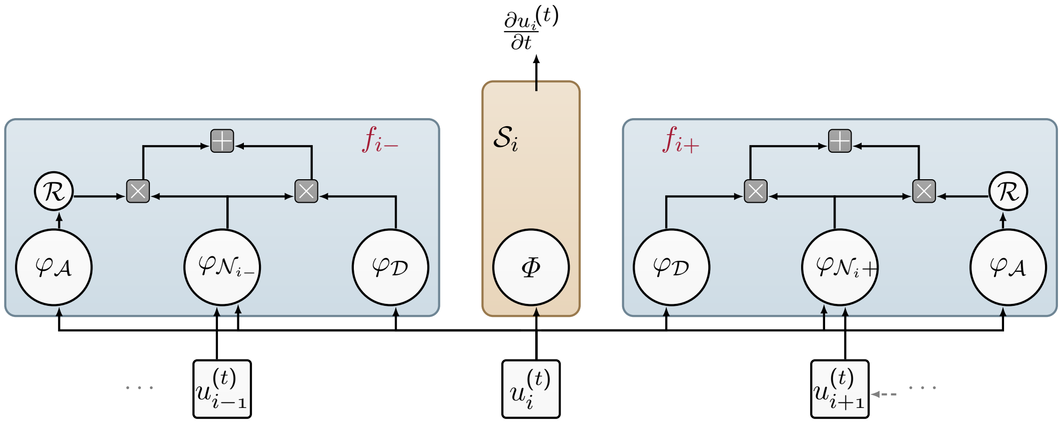

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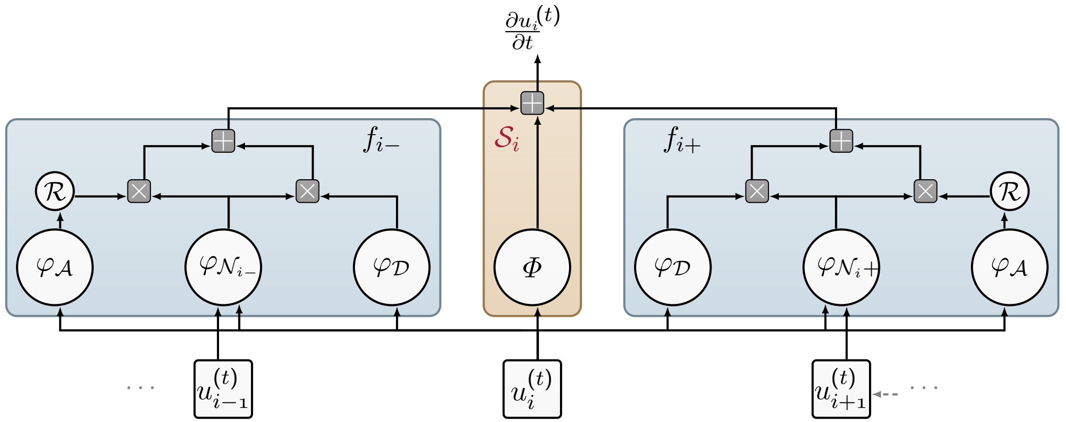

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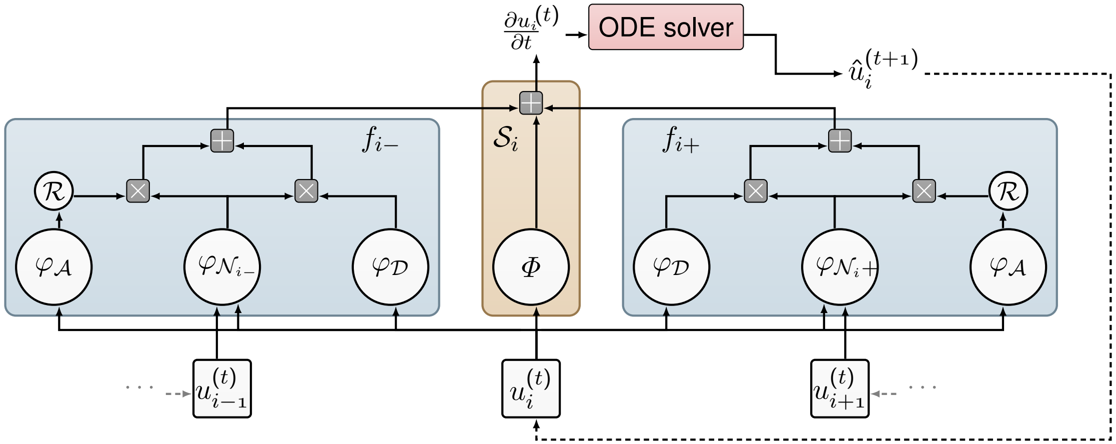

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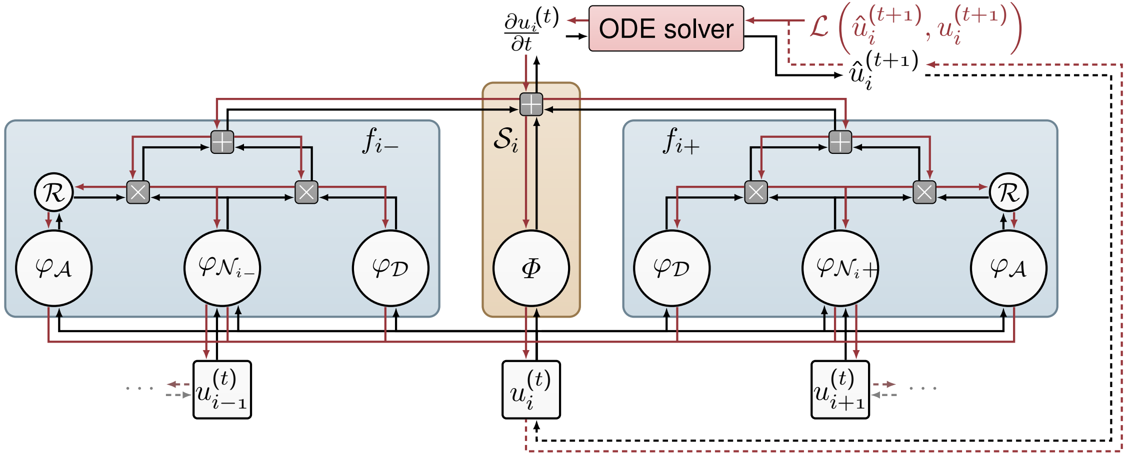

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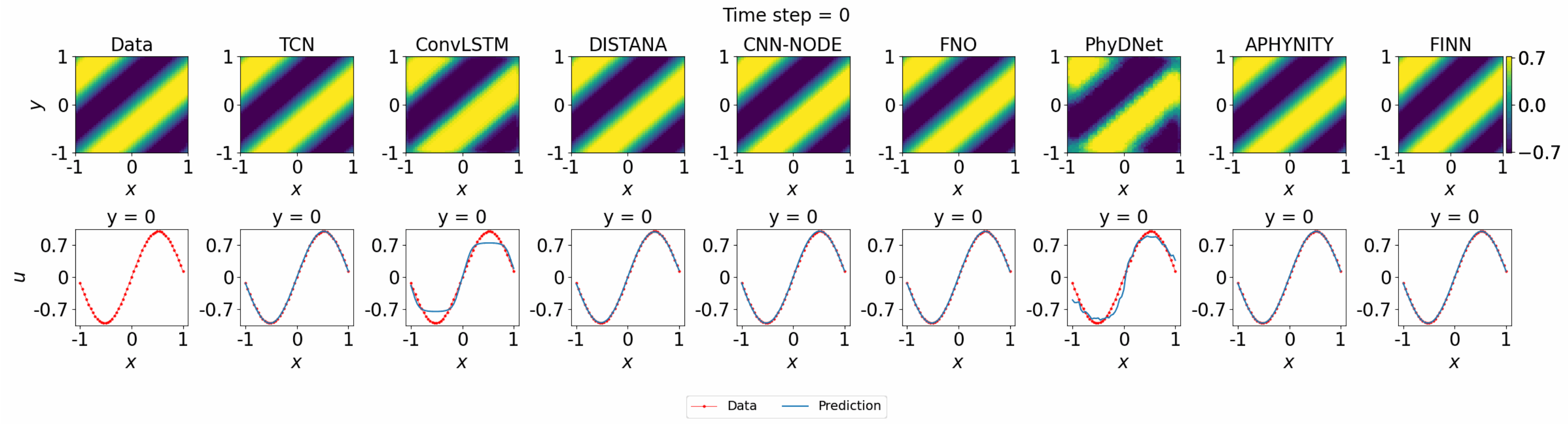
Benchmark: 2D Burgers' equation



Applications

Ongoing work

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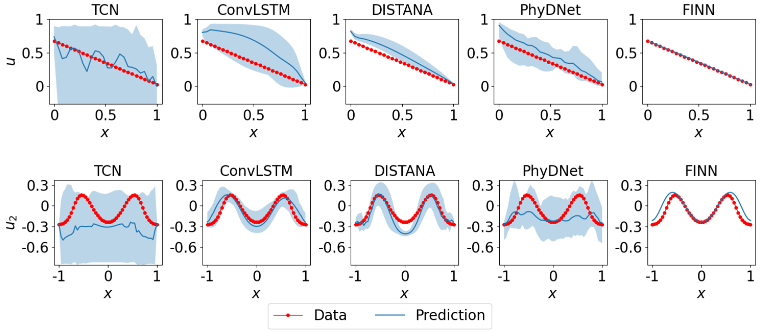
Ongoing work

- **Characterization** of **substance diffusion**



Applications

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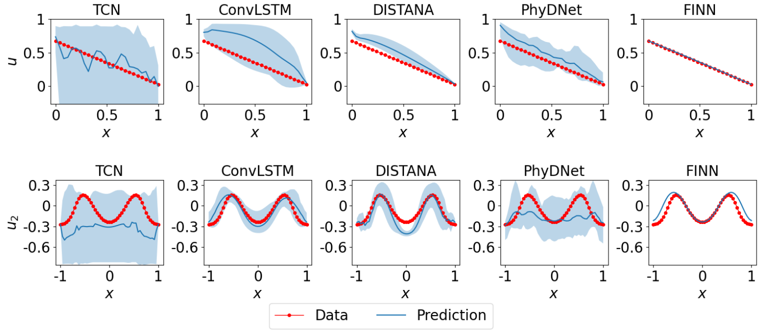
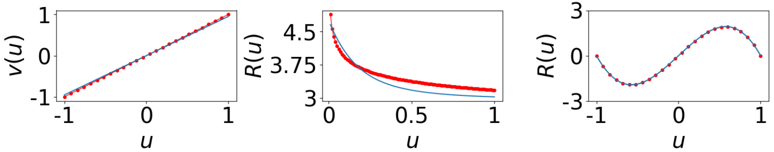


- **Determination** of **unknown functions**



Applications

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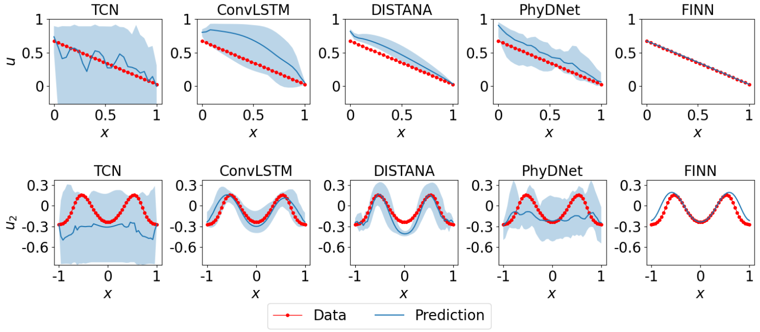

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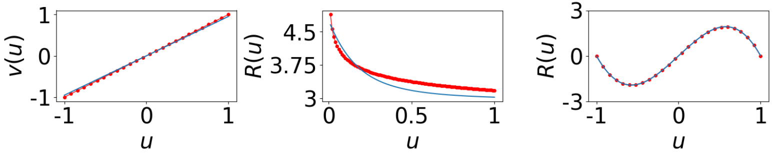
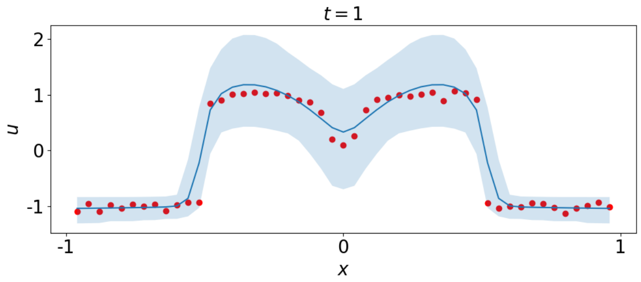


Ongoing work

▶ **Direct uncertainty estimates**



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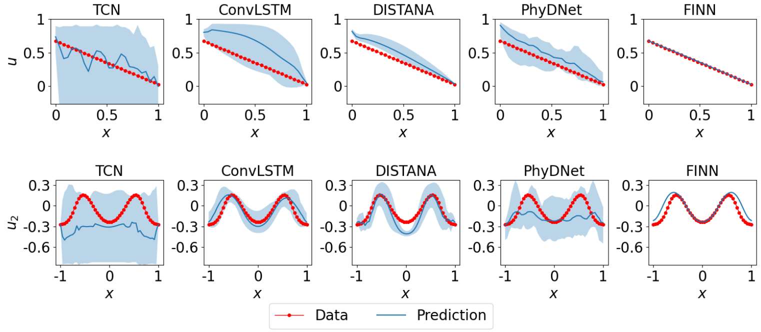

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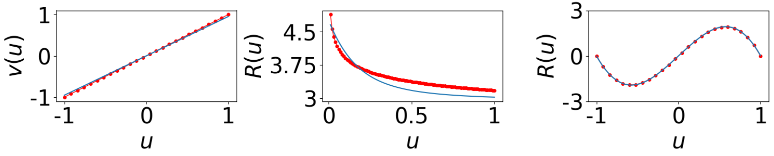

Ongoing work

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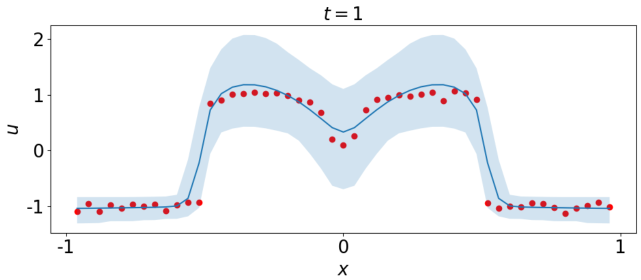

▶ **Inference** of **boundary conditions**