# Towards Practical Mean Bounds for Small Samples

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# Non-parametric Upper-Confidence Bound

**Assumption.**

- The distribution has support $\mathcal{D} \subseteq [0, 1]$.

**Definition.**

Given a confidence level $1 - \alpha \in (0, 1)$, an upper confidence bound $\mu_{upper}^{1-\alpha}$ has *guaranteed coverage* if, for all sample sizes $1 \leq n \leq \infty$ and for all distributions $F$ with support on $[0, 1]$, it satisfies

$$Prob_F[\mu \leq \mu_{upper}^{1-\alpha}(X_1, X_2, ..., X_n)] \geq 1 - \alpha, \tag{1}$$

where $\mu$ is the mean of the unknown distribution $F$.

# Existing Results

Existing confidence bounds for the mean of distributions with support $\mathcal{D} \subseteq [0, 1]$.

- With guarantees, but bad performance for small samples (Maurer-Pontil, Hoeffding and Anderson)
- No guarantees (assume Gaussianity), good performance for small samples (Student-t)

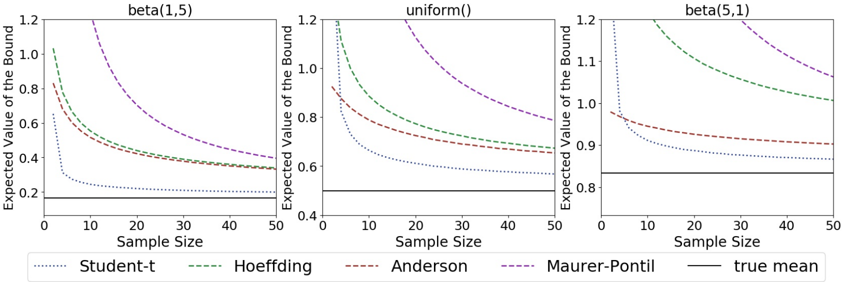


**Figure 1:** The expectation of the confidence bounds for small samples. .

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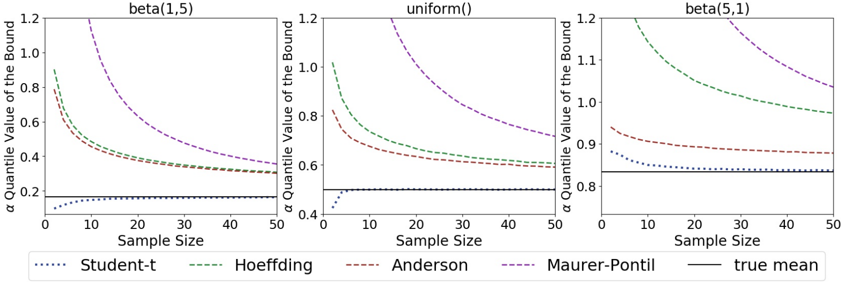


**Figure 2:** The $\alpha$-quantile of the confidence bounds for small samples. Student-t does not have guarantee for small sample size.

# Motivations

Consider the problem of finding mean bounds for **small** sample sizes. Examples:

- Phase 1 Clinical Trial.
- Safe Reinforcement Learning: importance-weighted estimators are used to estimate the policy's return.

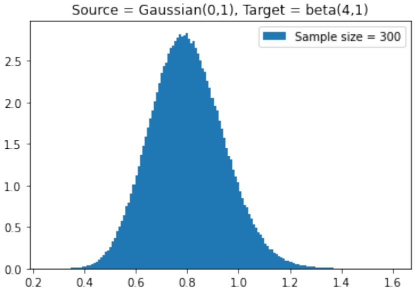


**Figure 3:** The sample mean of importance-weighted estimator is skewed even for large sample size.

# Problem Statement

Find an upper confidence bound:

- with guaranteed coverage for distributions on $[0, 1]$
- with good performance for small sample size

# Our Bound

- Input:
  - A sample $\mathbf{x} = (x_1, \cdots, x_n)$ of size $n$.
  - A confidence level $\alpha$.
  - A function $T : [0, 1]^n \to R$.
    - Example: $T(\mathbf{x})$ is the sample mean of $\mathbf{x}$.
- Output: upper-confidence bound $b^{\alpha}_{T}(\mathbf{x})$.
- Our bound is parameterized by a function $T$. Function $T$ defines the order of the bound: If $T(\mathbf{x}) \leq T(\mathbf{y})$ then $b^{\alpha}_{T}(\mathbf{x}) \leq b^{\alpha}_{T}(\mathbf{y})$.
  - Note: Hoeffding's bound and Student-t's bound order the samples by the sample mean.

# Theoretical Results

A new bound with guarantee coverage and good performance:

- For any sample size, for any $\alpha$, for any sample $\mathbf{x}$, our bound is smaller than or equal to Anderson's [Anderson, 1969], one of the best bound for small sample sizes.
  - For any sample size, for any $\alpha$, for any sample $\mathbf{x}$, Anderson's bound is smaller than or equal to Hoeffding [Hoeffding, 1963].
- Therefore for any sample size, for any $\alpha$, for any sample $\mathbf{x}$, our bound is smaller than or equal to Hoeffding's.

# Simulations i

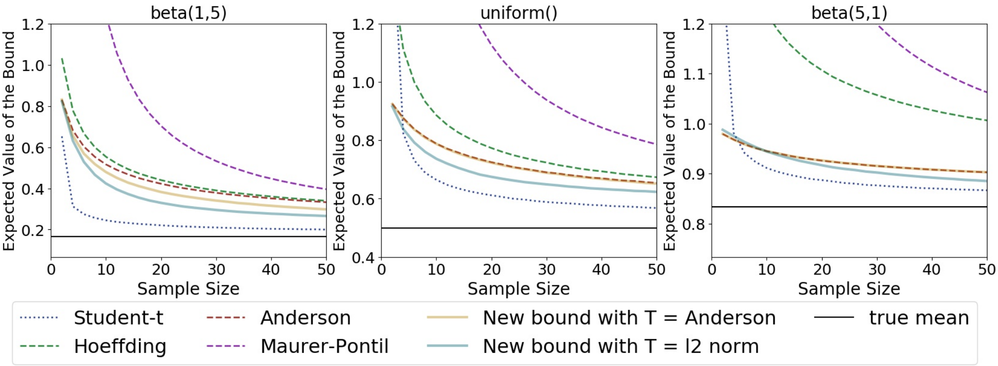


**Figure 4:** Expected values of the bounds versus sample size $n$. For each $n$, we sample $\mathbf{X} \in \mathcal{R}^n$ 10,000 times, and take the average of the bound. Our new bound has better performance than Anderson's, Hoeffding's and Maurer and Pontil's.

# Simulations ii

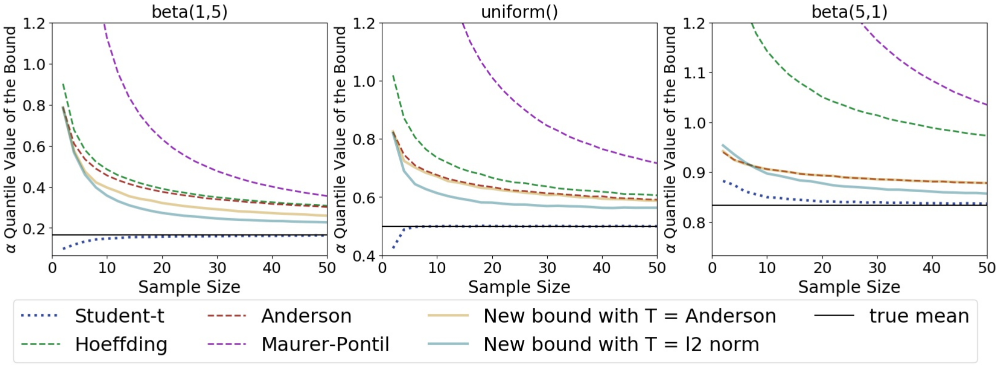


**Figure 5:** The $\alpha$-quantiles of bound distributions. If $\alpha$ portion of the samples' bound is below the true mean, the bound does not have guarantee. For the $\mathtt{uniform}(0, 1)$ and $\mathtt{beta}(1, 5)$ distribution, when the sample size is small, Student-t does not have guarantee.

# Thank You!

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# References

T. W. Anderson. Confidence limits for the value of an arbitrary bounded random variable with a continuous distribution function. *Technical Report Number 1, Department of Statistics, Stanford University*, 1969.

W. Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.