Two-way kernel matrix puncturing: towards resource-efficient PCA and spectral clustering

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June 11, 2021

"One-way" kernel puncturing: Improving over subsampling

- **Setting**: $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$, $\mathbb{R}^p \ni \mathbf{x}_i \sim \mathcal{N}(\pm\boldsymbol{\mu}, \mathbf{I}_p)$, $p, n \to \infty$, $p/n \to c_0$
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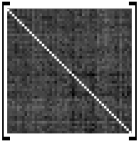

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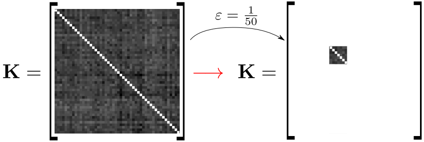

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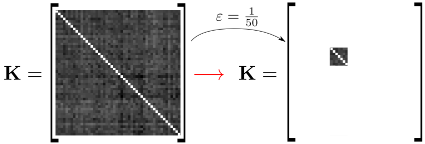

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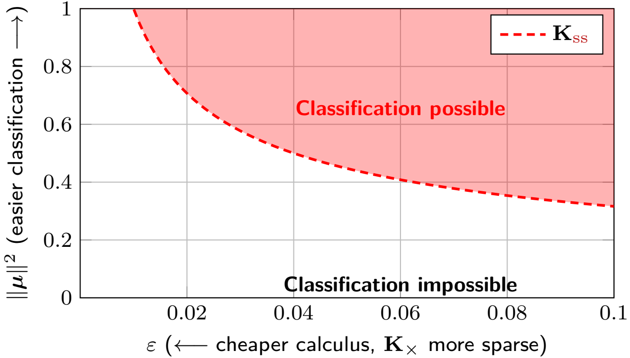

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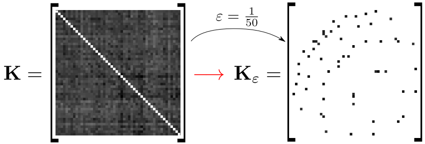

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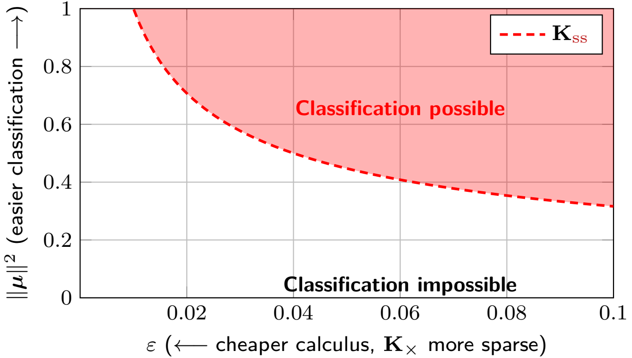

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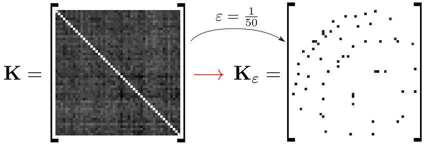

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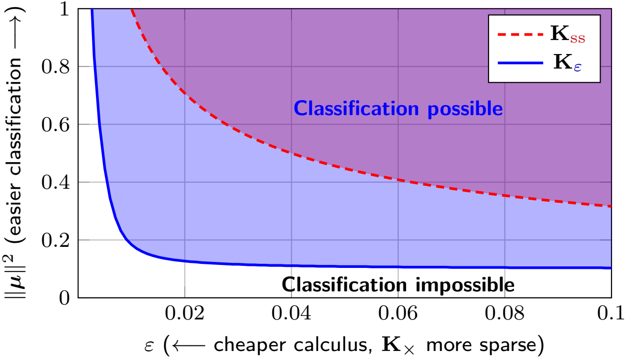

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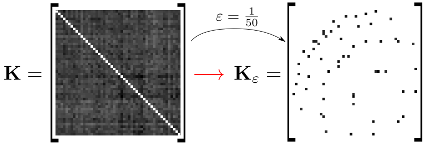

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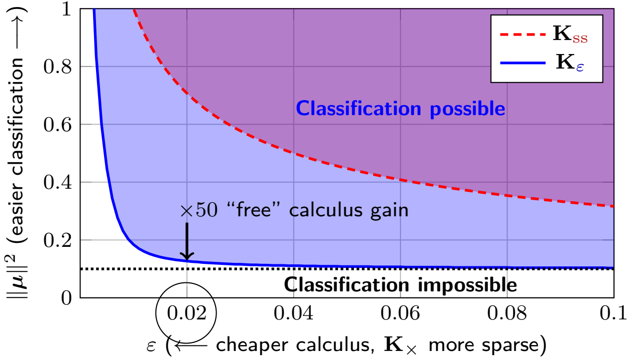

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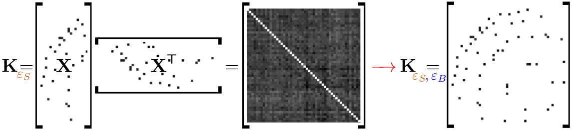

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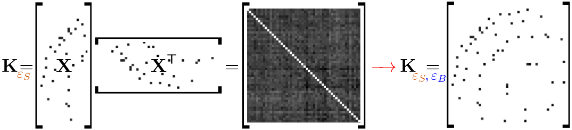

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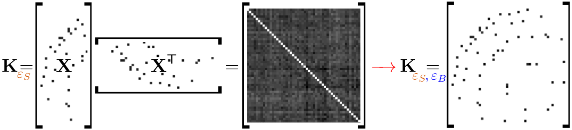

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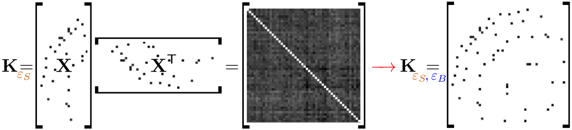

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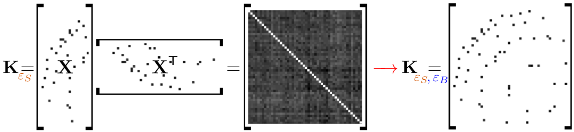

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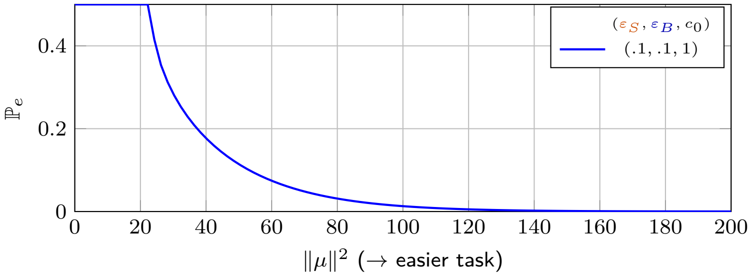

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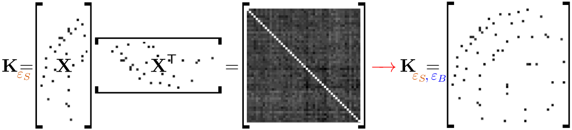

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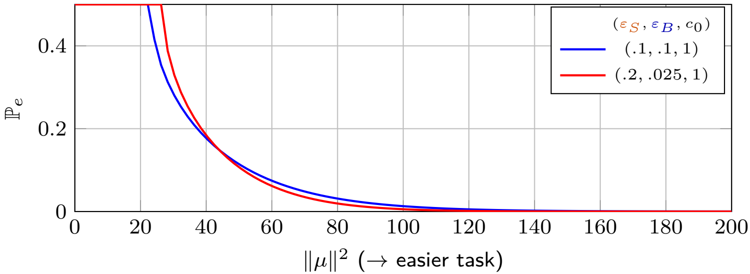

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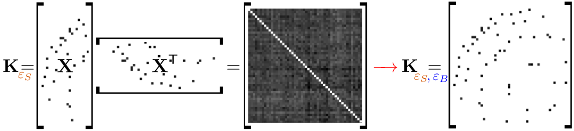

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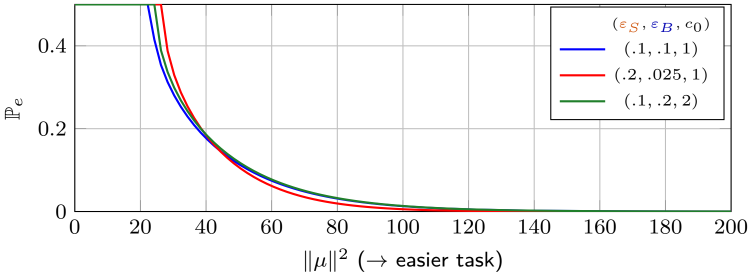

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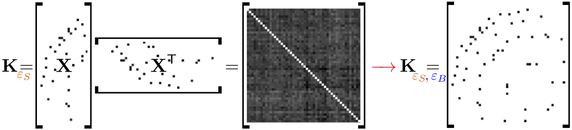

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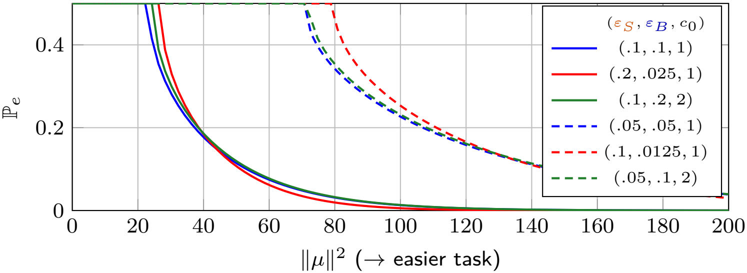
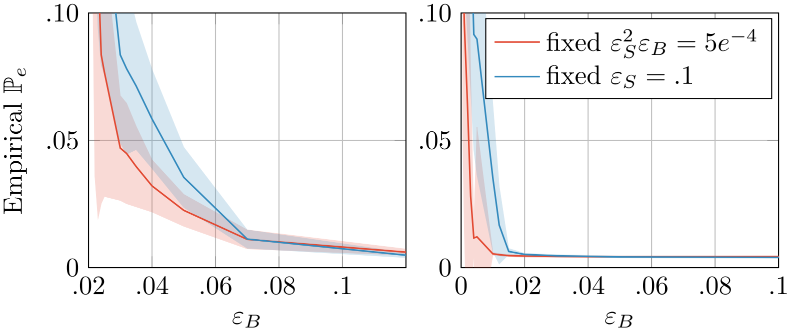
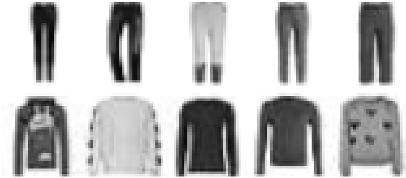
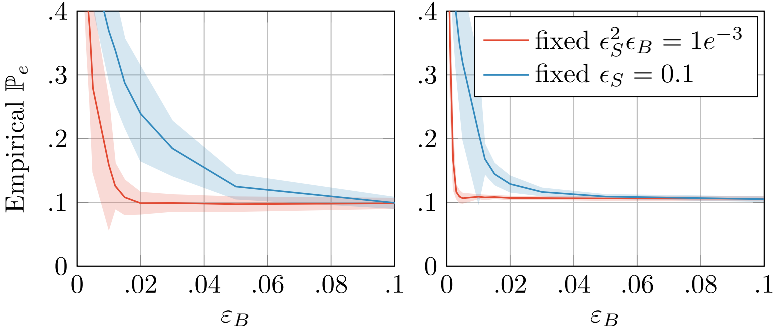


➽ **BigGAN images ('tabby' vs 'collie'):** VGG features, $p = 4\,096$, $n = 2\,500$ (**left**), $n = 10\,000$ (**right**)



➽ **MNIST-fashion ('trouser' vs 'pullover')**: $p = 784$, $n = 512$ (**left**), $n = 2048$ (**right**)





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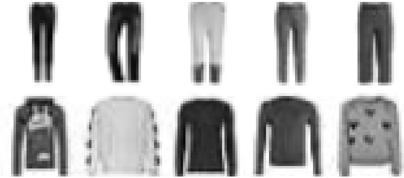

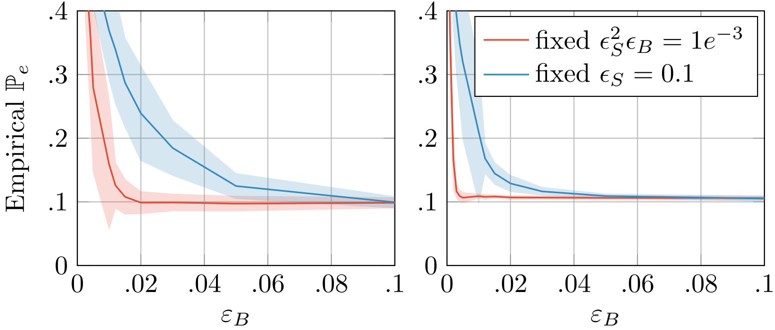

Thank You.