Interpretable Stability Bounds for Spectral Graph Filters

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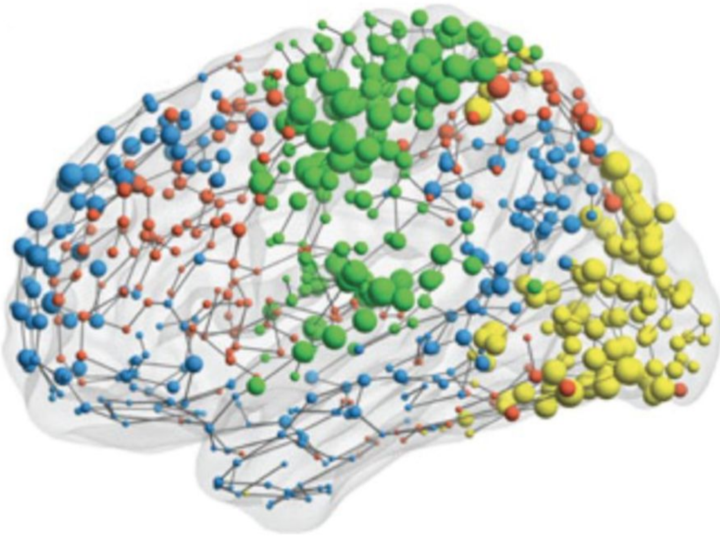
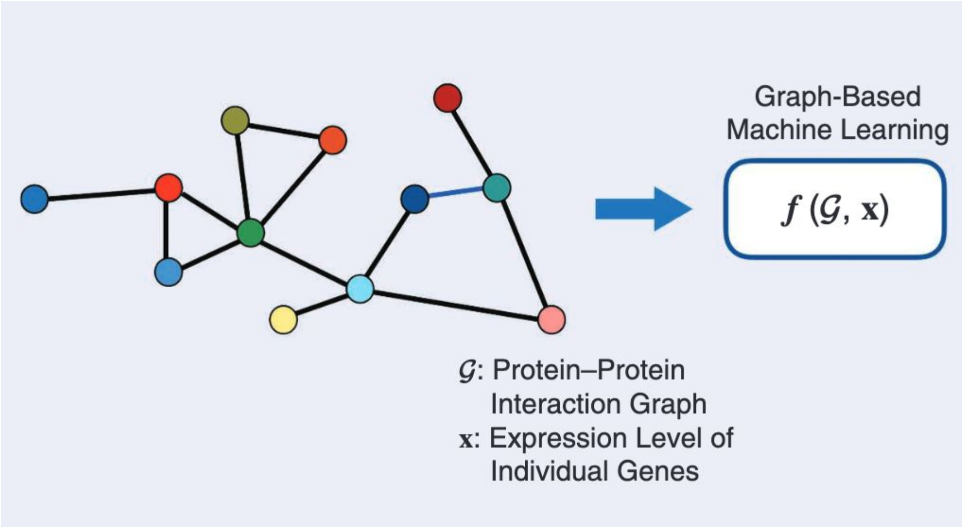
Introduction

Graph Structured Data



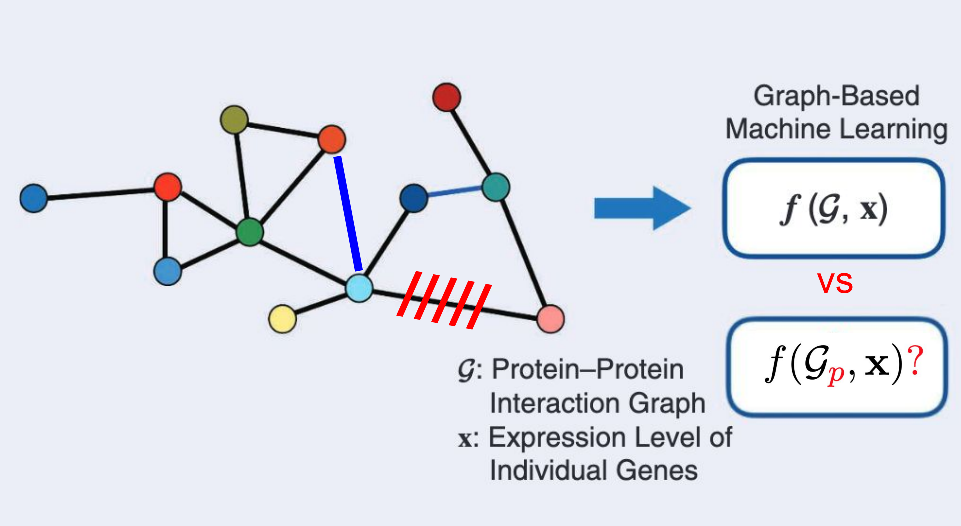
Graph Based Models

- Must take into account the data (signal) on each node
- As well as the domain for which the data resides (graph)



Stability

- What happens to the output of the model if the domain changes
- Why is it important?
  - Noisy/unreliable graph
  - Adversaries
  - Transferability
  - Changes through time



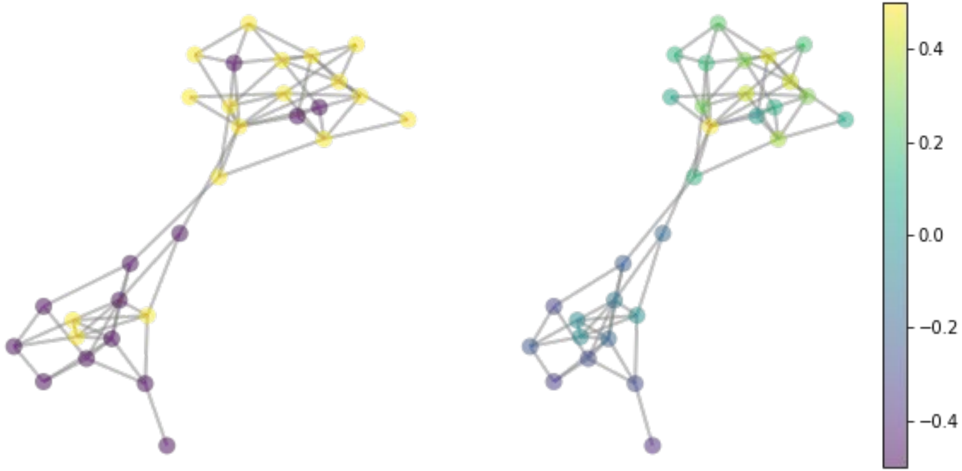
Stability of spectral graph filters

Spectral graph filters

- Amplifies or attenuates frequencies of signals on the graph
- Can be defined as a function operating on the normalised Laplacian matrix

Example of a low-pass (smoothing) filter:

$$\mathbf{y} = g(\mathbf{L})\mathbf{x}$$
$$= (\mathbf{I}_n + \mathbf{L})^{-1}\mathbf{x}$$



Stability

The low-pass filter

$$g(\mathbf{L}) = (\mathbf{I}_n + \mathbf{L})^{-1}$$

Satisfies the following stability property

$$\frac{\|g(\mathbf{L})\mathbf{x} - g(\mathbf{L}_p)\mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq \|\mathbf{E}\|_{\text{op}}\,, \qquad \mathbf{E} = \mathbf{L}_p - \mathbf{L}$$

Where $\mathbf{L}_p$ is the normalised Laplacian of the graph after perturbation

We will refer to $\|\mathbf{E}\|_{\text{op}}$ as the error norm

Interpretable bounds on the error norm

It is not easy to reason when the error norm will be small or large

We introduce the following bound

$$\frac{\|g(\mathbf{L})\mathbf{x} - g(\mathbf{L}_p)\mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq \max_{u \in \mathcal{V}} \left\{ \frac{\Delta_u^-}{\sqrt{d_u \delta_u}} + \frac{\Delta_u^+}{\sqrt{d'_u \delta'_u}} + \left(\frac{\alpha_u}{1 - \alpha_u} \right) \frac{d_u - \Delta_u^-}{\sqrt{d_u \delta_u}} \right\}$$

Right hand side is small when

- We perturb between high degree nodes
- The perturbation is distributed

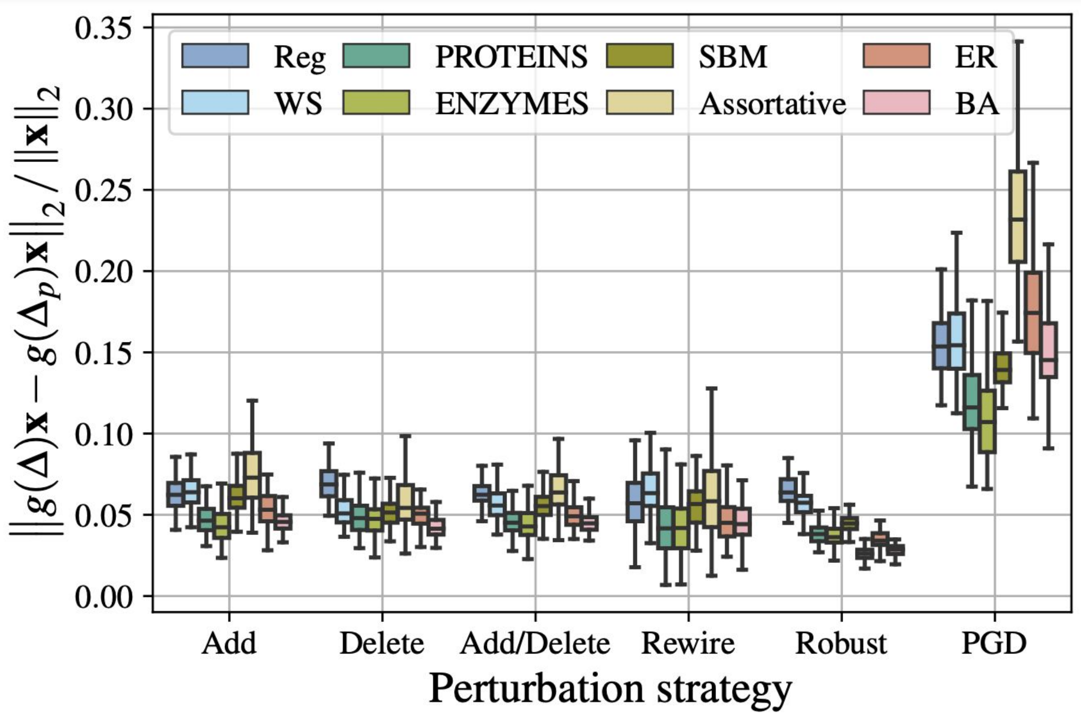
Experiments

Experiment overview

- We consider many random graph models and real-world graphs
- We perturb with a variety of strategies
  - Random strategies
  - Adversarial strategy
  - Robust strategy based on our bound
- Experimentally test looseness of our bound
- Experimentally test how stable filtering is to perturbation

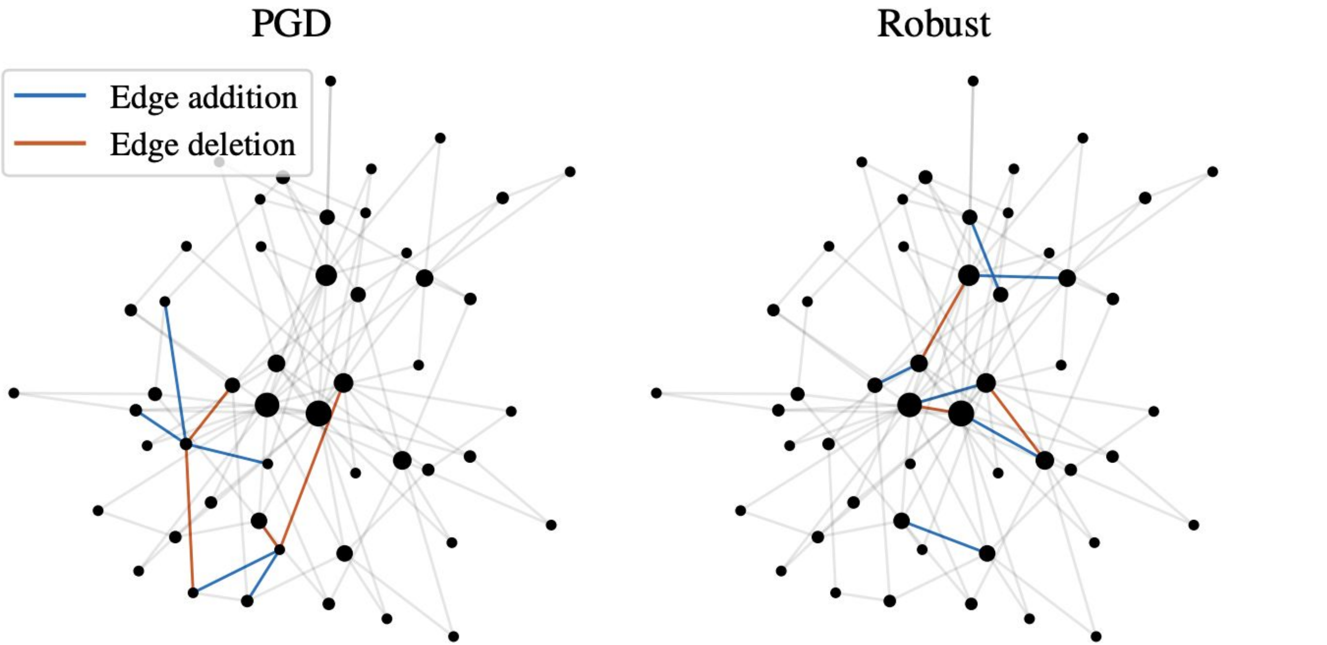
Relative output change

- Largest change when perturbing using adversarial attack strategy
- Least change when using a Robust strategy, based on our bound



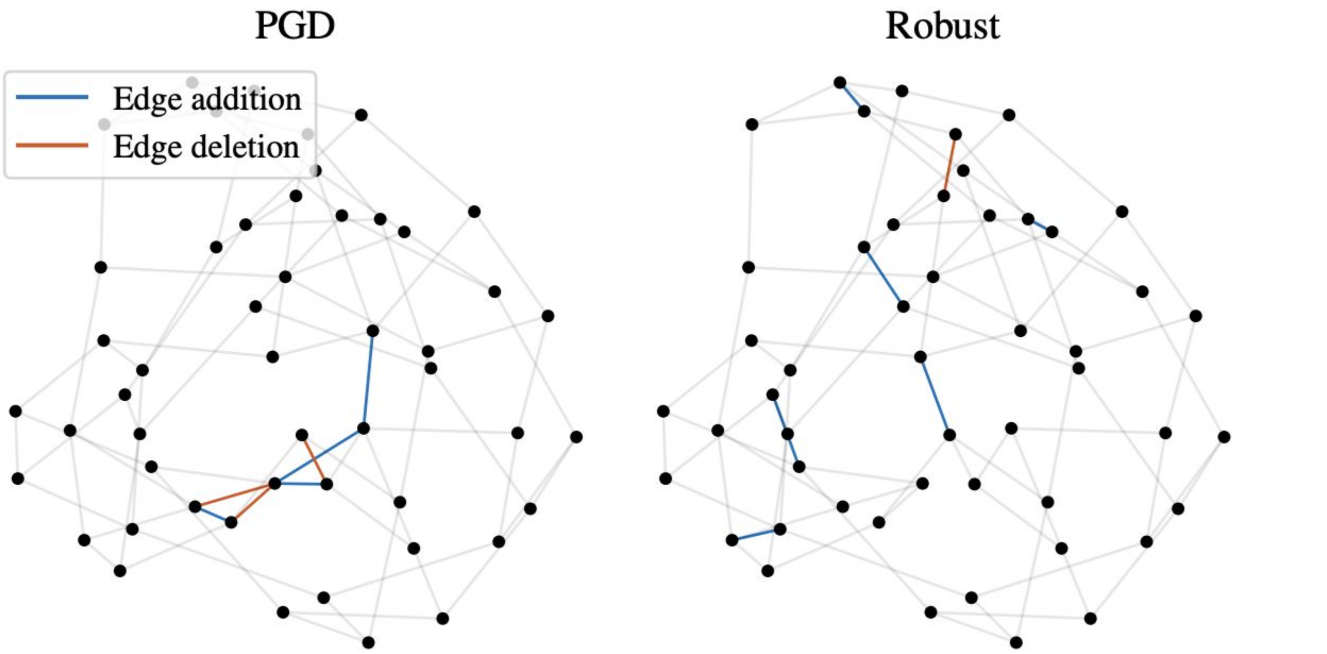
Qualitative Analysis of non-random strategies

Can observe the role of node degree when perturbing Barabasi Albert graphs



Qualitative Analysis of non-random strategies

Can observe the role of perturbation concentration when perturbing K-regular graphs



Future work

- Quantitative/statistical analysis of the role of degree and locality
- Comparison with graph neural networks

Thank you

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