Random Matrix Improved Covariance Estimation for a Large Class of Metrics

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Context

Observations:

- $X = [x_1, \ldots, x_n]$, $x_i \in \mathbb{R}^p$ with $\mathbb{E}[x_i] = 0$, $\mathbb{E}[x_i x_i^\mathsf{T}] = C$.

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- Numerical inversion of asymptotic spectrum (QuEST).
  1. Bai-Silverstein equation: Estimate $\lambda(\hat{C})$ from $\lambda(C)$ in "large p, n" regime.
  2. Need for non trivial inversion of the equation.

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- Elementary idea
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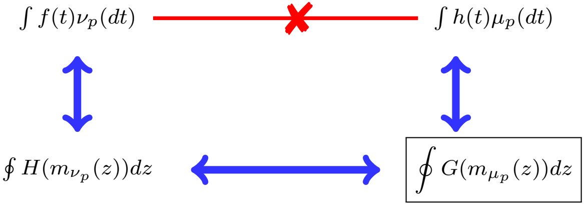

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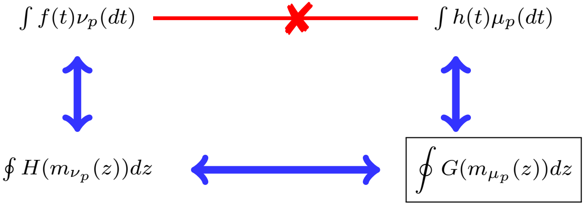

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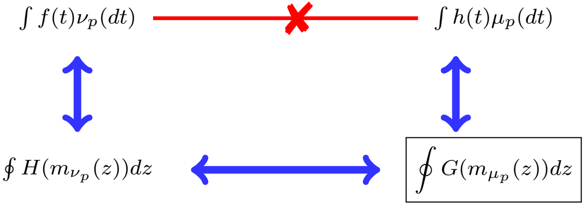

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- Proposed estimation

$$\boxed{\check{C} \equiv \operatorname{argmin}_{M \succ 0} h(M), \quad h(M) = \hat{\delta}(M, X)^2}$$

- Gradient descent over the Positive Definite manifold.

Algorithm 1 Proposed estimation algorithm.

Require $M_0 \in C_n^{++}$.

Repeat $M \leftarrow M^{\frac{1}{2}} \exp\left(-tM^{-\frac{1}{2}} \nabla h_X(M) M^{-\frac{1}{2}}\right) M^{\frac{1}{2}}$.

Until Convergence.

Return $\check{C} = M$.

Experiments

- 2 Data classes $x_1^{(1)}, \ldots, x_{n_1}^{(1)} \sim N(\mu_1, C_1)$ and $x_1^{(2)}, \ldots, x_{n_2}^{(2)} \sim N(\mu_2, C_2)$.
- Classify point x using Linear Discriminant Analysis based on the sign of

$$\delta_x^{\mathrm{LDA}} = (\hat{\mu}_1 - \hat{\mu}_2)^\mathsf{T} \check{C}^{-1} x + \frac{1}{2} \hat{\mu}_2^\mathsf{T} \check{C}^{-1} \hat{\mu}_2 - \frac{1}{2} \hat{\mu}_1^\mathsf{T} \check{C}^{-1} \hat{\mu}_1.$$

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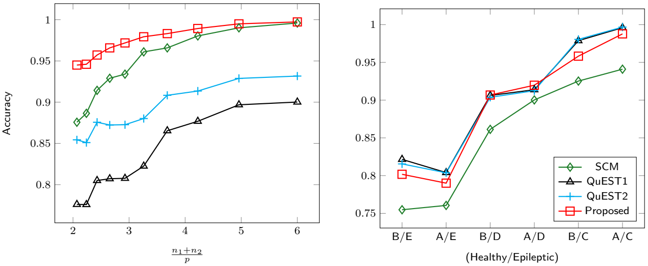

Figure: Mean accuracy obtained over 10 realizations of LDA classification. (Left) C_1 and C_2 Toeplitz-0.2/Toeplitz-0.4, and (Right) real EEG data.