On Dropout and Nuclear Norm Regularization

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Motivation

▶ Algorithmic approaches endow deep learning systems with certain inductive biases that help generalization.
▶ In this paper we study dropout, one of the most popular algorithmic heuristics for training deep neural nets.

Srivastava, Hinton, Krizhevsky, Sutskever and Salakhutdinov

(a) Standard Neural Net  (b) After applying dropout.
Problem Setup

- Deep linear networks with \( k \) hidden layers

\[
f_w : x \mapsto W_{k+1} \cdots W_1 x, \quad W_i \in \mathbb{R}^{d_i \times d_{i-1}}
\]

where \( w = \{W_i\}_{i=1}^{k+1} \) is the set of weight matrices.
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where $w = \{W_i\}_{i=1}^{k+1}$ is the set of weight matrices.

- $x \in \mathbb{R}^{d_0}$, $y \in \mathbb{R}^{d_{k+1}}$, $(x, y) \sim \mathcal{D}$. Assume $\mathbb{E}[xx^\top] = I$. 

![Diagram](image-url)
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- Learning problem: minimize the population risk

$$L(w) := \mathbb{E}_{(x,y) \sim \mathcal{D}}[\|y - f_w(x)\|^2]$$

based on iid samples from the distribution.
Network perturbed by dropping hidden nodes at random, computing

\[ \tilde{f}_w(x) = W_{k+1}B_k W_k \cdots B_1 W_1 x, \]

where \( B_i(j,j) = 0 \) with probability \( 1 - \theta \), and \( \frac{1}{\theta} \) with probability \( \theta \).
Problem Setup

- Network perturbed by dropping hidden nodes at random, computing

\[ \bar{f}_w(x) = W_{k+1}B_kW_k \cdots B_1W_1x, \]

where \( B_i(j,j) = 0 \) with probability \( 1 - \theta \), and \( \frac{1}{\theta} \) with probability \( \theta \).

- dropout boils down to SGD on the \textit{dropout objective}

\[ L_\theta(w) := \mathbb{E}_{\{B_i\},(x,y)} \| y - \bar{f}_w(x) \|^2 \]
Empirical Observation

- 3-layer network with width/input/output dimensionality $= 20$. 

![Graph showing singular values](image-url)
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![Graph showing singular values comparison between True Model and SGD.](image)
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Main Results

Explicit Regularizer
Give full characterization of $R(w) := L_\theta(w) - L(w)$
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$$\Theta(M) := \min_{f_w = M} R(w)$$
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- Multi-dimensional output

$\Theta^{**}(f_w) = \nu_{\{d_i\}} \|f_w\|_*^2$
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- **Multi-dimensional output**
  $$\Theta^{**}(f_w) = \nu\{d_i\} \|f_w\|_*^2$$

- **One-dimensional output**
  $$\Theta(f_w) = \Theta^{**}(f_w) = \nu\{d_i\} \|f_w\|^2$$
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Effective Regularization Parameter

$\nu_{\{d_i\}}$ increases with depth and decreases with width
deep and narrower networks are more biased towards low-rank solutions
Thanks for your attention!

Stop by Poster 79 for more information.