Lexicographic and Depth-Sensitive Margins in Homogeneous and Non-Homogeneous Deep Models

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ICML, 2019
Motivation

- Deep neural networks have multiple global minima.
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Main Goal

We would like to understand the minima selection process in training deep neural networks.
Empirical loss:

\[ \min_{\theta} \mathcal{L}(\theta) = \sum_{n=1}^{N} e^{-f_n(\theta)} \]

- \( f_n(\theta) \) - the prediction function,
- \( N \) - number of samples.
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The inductive bias introduced in our learning process affects which specific global minimizer is chosen.
Inductive Bias Sources

1) Regularization path:

\[ \Theta_r(\lambda) = \arg\min_{\theta} \mathcal{L}(\theta) + \lambda \|\theta\|_2^2 \]  

(1)

Empirically, using small, and even vanishing \( \lambda \) can improve generalization.

What happens at the limit of the regularization path, when \( \lambda \to 0 \)?

2) Constrained path:

\[ \Theta_c(\rho) = \arg\min_{\theta} \mathcal{L}(\theta) \text{ s.t. } \|\theta\|_2 \leq \rho \]

Previously related to problem (1).

What happens at the limit of the constrained path, when \( \rho \to \infty \)?

3) Optimization path:

\[ \bar{\theta}(t) = \theta(t) \|\theta(t)\|, \quad \Delta \theta(t) = -\eta \nabla \mathcal{L}(\theta(t)) \]

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Previous Results

• For linear prediction functions:
  ▶ Optimization path $\Rightarrow$ Max-Margin solution.

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- For homogeneous prediction functions, e.g., ReLU networks:
  - Regularization path $\Rightarrow$ Max-Margin solution.

We study how infinitesimal regularization or gradient descent optimization lead to margin maximizing solutions in both homogeneous and non-homogeneous models.
Main Contributions - Non-Homogeneous Models

- For $f_n(\theta) = \text{sum of homogeneous functions of different orders}$: we characterized the constrained path asymptotic solution.
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**Implication**

In an ensemble of homogeneous neural networks, e.g., feedforward ReLU networks, the ensemble will aim to discard the most shallow network.
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A: Yes, we find general conditions under which the optimization path converges to:
   1) stationary points of the constrained path.
   2) max-margin solutions.
Refined characterization:

- For non-convex prediction functions the max-margin solution is not necessarily unique.
- We show that the constrained path converges to a specific type of max-margin solution.
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Q: Is margin maximization all that we do?
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  - For non-convex prediction functions the max-margin solution is not necessarily unique.
  - We show that the constrained path converges to a specific type of max-margin solution.

Q: Is margin maximization all that we do?

A: No. After maximizing the distance to the closest data point (max-margin), we also maximize the distance to the second closest data point, and so on.
Thank You!

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