Learning to optimize multigrid PDE solvers

DANIEL GREENFELD, WEIZMANN INSTITUTE OF SCIENCE

JOINT WORK W. MEIRAV GALUN, RON KIMMEL, IRAD YAVNEH AND RONEN BASRI
Solving PDEs is useful

- Predicting weather systems
- Aircraft and auto design
- Oceanic flow
Solving PDEs is hard

- High accuracy requires discretization on very fine grids
- Developing efficient solvers is an active research area since many decades ago
- Can we use machine learning to construct solvers?
Previous works

Learning to solve a single equation (new equation = retrain needed)

- Katrutsa et al, 2017: learning the prolongation for Poisson equation
- Hsieh, 2019: accelerate Poisson solvers
- Baque et al, 2018: simulate fluid dynamics
- Han et al, 2018: PDEs in high dimension
- ...
This work

- Learning how to solve a family of PDEs
- Example: 2D elliptic diffusion problems
  \[-\nabla \cdot (g \nabla u) = f\]
- Focus on *multigrid solvers*
  - Solves the equation on multiple scales
  - Prolongation operator for moving between scales
Key elements of our approach

- **Scope** - train a single network once for an entire class of PDEs

- **Unsupervised training** - no ground truth provided, and no equation is solved during training

- **Generalization** - train on small problems w. periodic BC & test on much larger problems w. Dirichlet BC

- **Efficient training** – using Fourier analysis
TL;DR

- We pose the following learning problem

\[
\min_{\theta} \mathbb{E}_{A \sim D} \rho \left( M(A, P_{\theta}(A)) \right)
\]

- \( \rho \left( M(A, P_{\theta}(A)) \right) \) measures the convergence rate of the solver

- \( P_{\theta}(A) \) is a NN mapping PDEs (discretization matrices) to multigrid solvers (prolongation operators)

- \( A \sim D \) is a distribution over PDEs (for example, a distribution over \( g \) in \( -\nabla \cdot (g \nabla u) = f \))
# Some results

<table>
<thead>
<tr>
<th>Grid size</th>
<th>V cycle</th>
<th>W cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>32x32</td>
<td>83%</td>
<td>100%</td>
</tr>
<tr>
<td>64x64</td>
<td>92%</td>
<td>100%</td>
</tr>
<tr>
<td>128x128</td>
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<td>99%</td>
</tr>
<tr>
<td>1024x1024</td>
<td>83%</td>
<td>98%</td>
</tr>
</tbody>
</table>

![Plot](image.png)
If interested, come check out our poster @ Pacific Ballroom #249