Geometry & Symmetry in Short-and-Sparse Deconvolution

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Short-and-Sparse (SaS) Deconvolution

Signals containing **Short Repeated** Motifs:

SaSD: Find **Short** $a_0$ & **Sparse** $x_0$ from **Convolution** $y = a_0 \ast x_0$
Symmetric Solutions in SaSD

All **scaled** & **shifts** of \((a_0, x_0)\) are solutions

\[
y = \alpha s_{\ell}[a_0] \ast (1/\alpha) s_{-\ell}[x_0]
\]

To solve \(a_0\...\)

- Fix scale \(\|\hat{a}\|_2 = 1\)
- Accept every signed shift \(\hat{a} = \pm s_{\ell}[a_0]\) as solution
Algorithm: Approximate Bilinear Lasso

Natural, effective algorithm: **Bilinear Lasso**

$$\min_{a \in \mathbb{S}^{p-1}, x \in \mathbb{R}^n} \lambda \|x\|_1 + \frac{1}{2} \|a \ast x - y\|_2^2$$

Theory: study **Approximate Bilinear Lasso**

$$\min_{a \in \mathbb{S}^{p-1}} \left( \min_{x \in \mathbb{R}^n} \lambda \rho(x) + \frac{1}{2} \|x\|_2^2 + \langle a \ast x, y \rangle \right)$$

$$=: \min_a \varphi_{ABL}(a) \quad s.t. \quad a \in \mathbb{S}^{p-1}$$

Here, $\rho$ is smoothed $\ell^1$ function

$\mathbb{S}^{p-1}$ is $p$-dimensional sphere
Geometry of Approximate Bilinear Lasso

**Over subspace** $S_{\{\ell_1, \ldots, \ell_3\}}$ spanned by shifts:

- **Local minimizers** are near shifts
- **Negative curvature** breaks symmetry between shifts

![Diagram](image)
Geometry of Approximate Bilinear Lasso

**Geometry of** $\varphi_{\text{ABL}}$ **is** **Benign** **over** **Union of Subspaces**
When is SaSD Easy?

**Shift–Coherence** $\mu$ of $a_0$:

$$\mu = \max_{i \neq j} |\langle s_i[a_0], s_j[a_0] \rangle|$$

**Sparsity** $\theta$ of $x_0$:

$$x_0 \sim \text{Bernoulli–Gaussian}(\theta)$$

SaSD is **harder** if...

- **Coherence** $\mu \uparrow$ (solutions closer on sphere)
- **Sparsity** $\theta \uparrow$ (more unknowns)
When is SaSD Easy?

**Sparsity–Coherence Tradeoff:**

- **Spiky**: $a_0$, $\mu \approx 0$, $\theta \approx p_0^{-1/2}$
- **Generic**: $\mu \approx p_0^{-1/2}$
- **Tapered generic lowpass**: $\mu \approx \text{const.}$

If $\mu$ of $a_0$ increases from $0 \uparrow 1$, then $\theta$ of $x_0$ decreases from $\frac{1}{\sqrt{p_0}} \downarrow \frac{1}{p_0}$.
**Thm1: Geometry of $\varphi_{ABL}$ over subspaces**

Given $a_0 \in \mathbb{R}^{p_0}$, $\mu$-shift coherent; $x_0 \sim \textrm{BG}(\theta)$ long and

$$\frac{1}{p_0} \lesssim \theta \lesssim \frac{1}{p_0 \sqrt{\mu} + \sqrt{p_0}},$$

then the local minima of $\varphi_{ABL}$ over UoS are close to shifts.

**Thm2: Provable algorithm for SaSD**

A minimizing algorithm starts and stays near a subspace, solves SaSD exactly up to a signed shift in poly time.
Wrapping Up

Main theoretical results: geometry of objective landscape, and a provable algorithm for SaSD.

Optimizing $\varphi_{ABL}$ is not recommended in practice.

Algorithmic ideas (sphere, initialization, etc.) are useful for practical method such as bilinear Lasso.
THANK YOU!