Generalized Majorization-Minimization





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Majorization Minimization (or Minorization Maximization)

- An iterative framework for non-convex optimization
- Examples of MM algorithm:
  - Expectation Maximization (EM)
  - Convex Concave Procedure (CCP)

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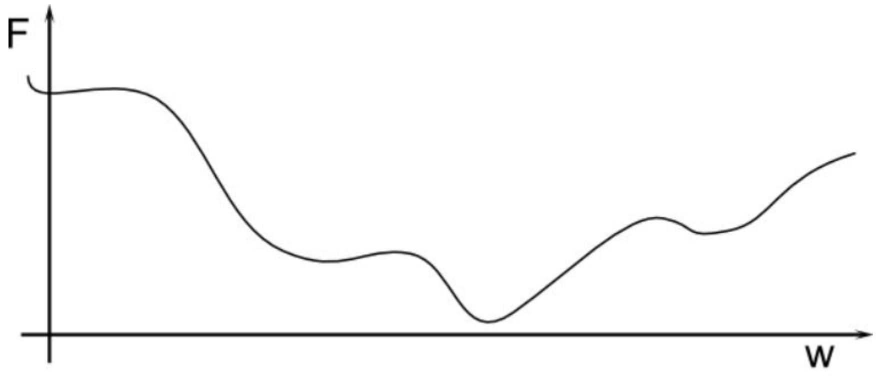

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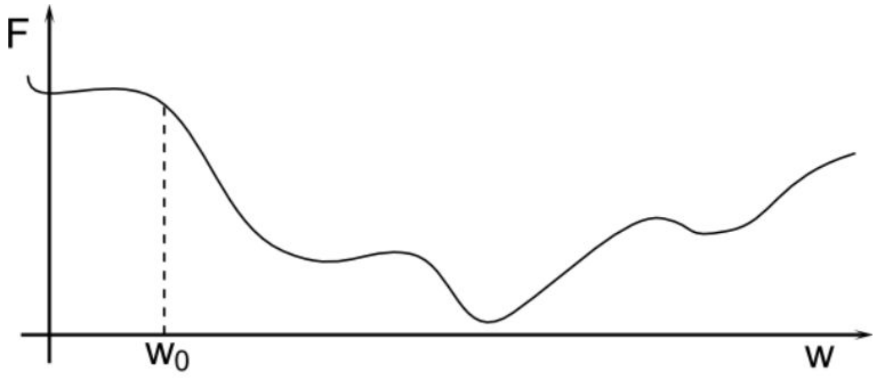

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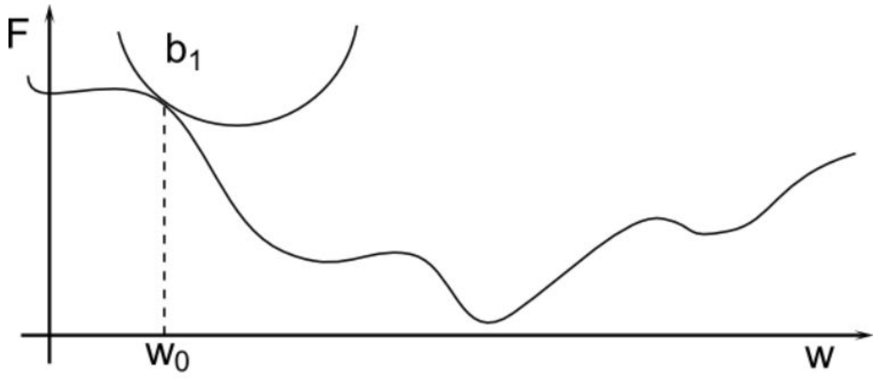

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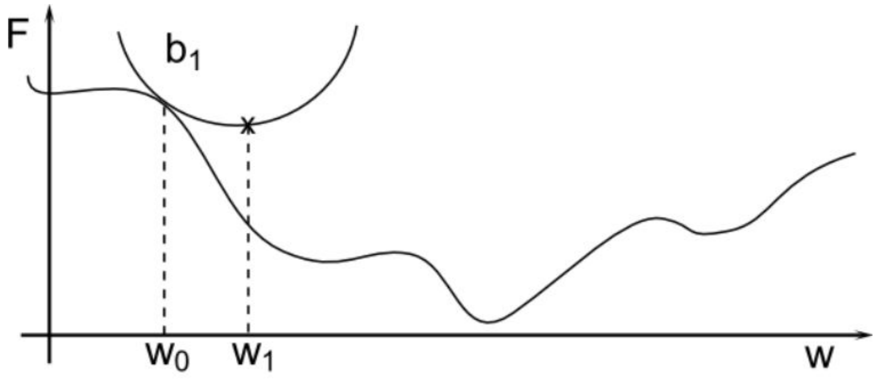

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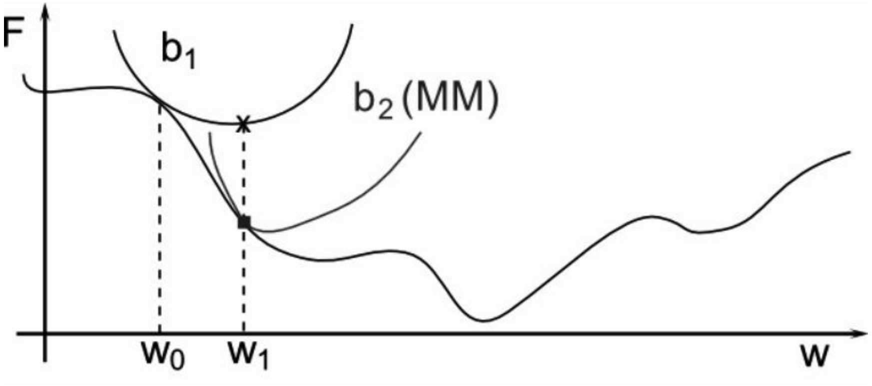

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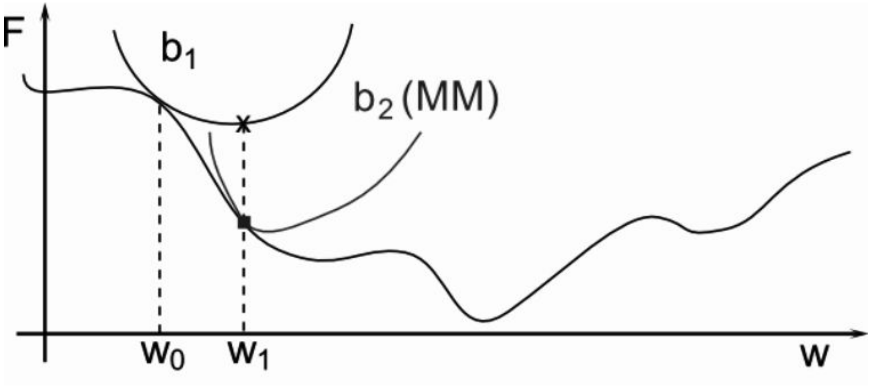

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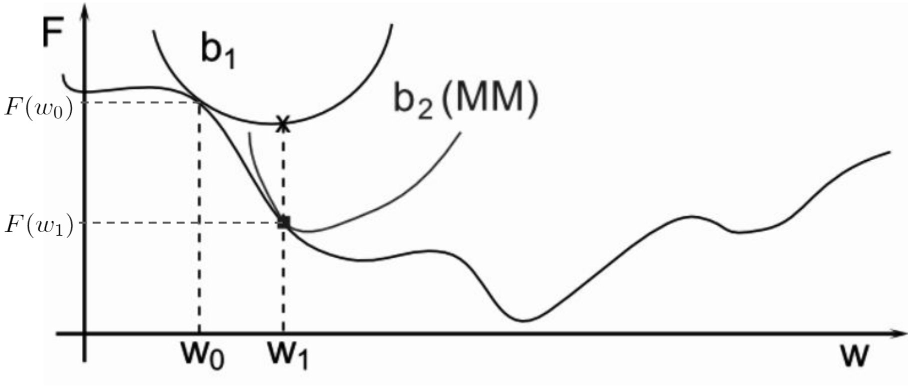

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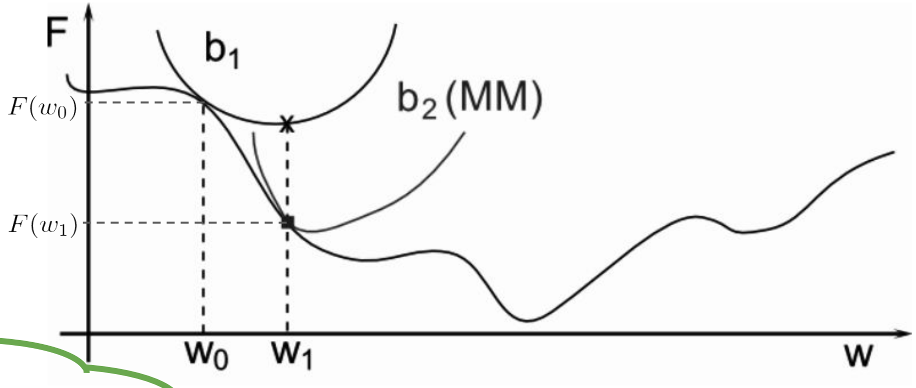

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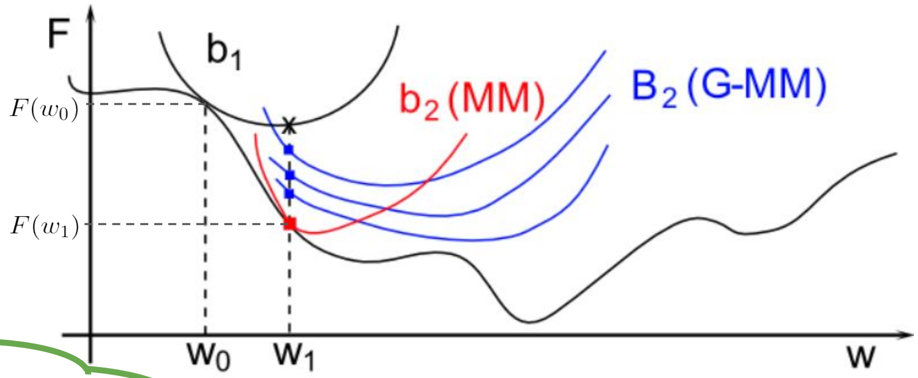

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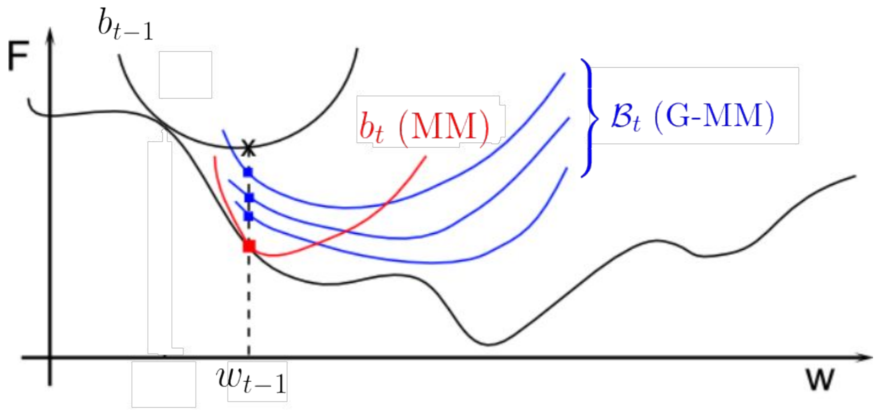


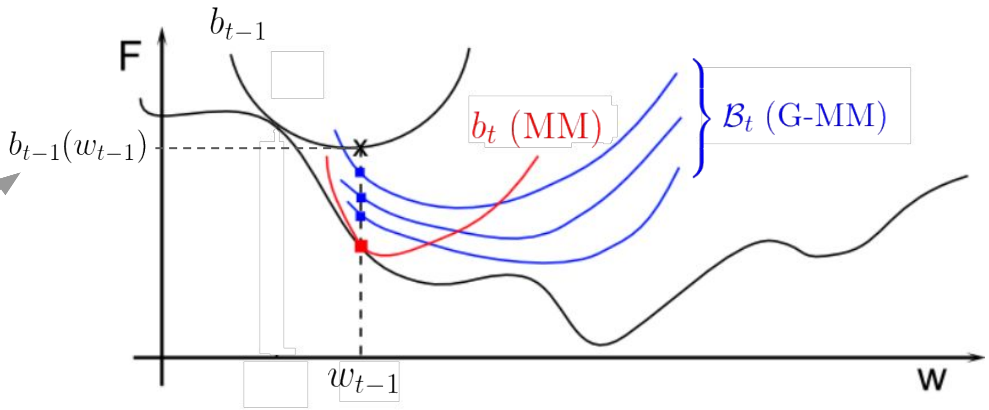
Bound selection

$\mathcal{F}$: family of bounds

$$\mathcal{B}(w, v) = \{b \in \mathcal{F} \mid b(w) \leq v\}$$

$$\mathcal{B}_t = \mathcal{B}(w_{t-1},\ b_{t-1}(w_{t-1}))$$

valid bounds
at iteration t



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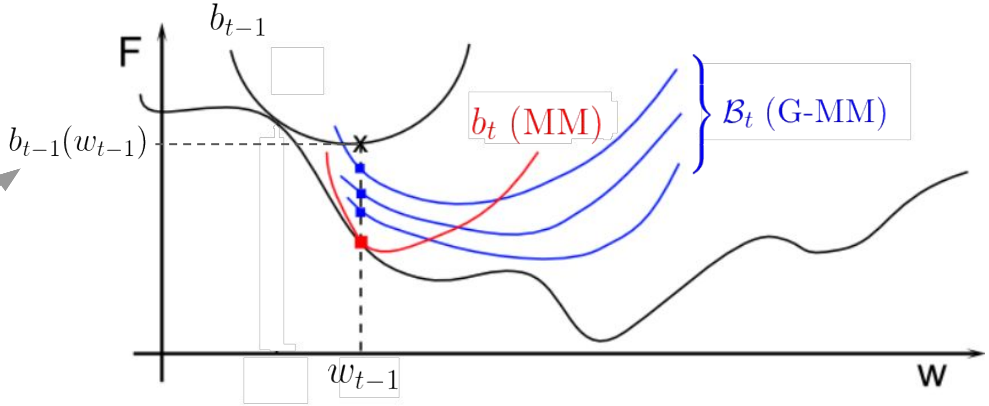

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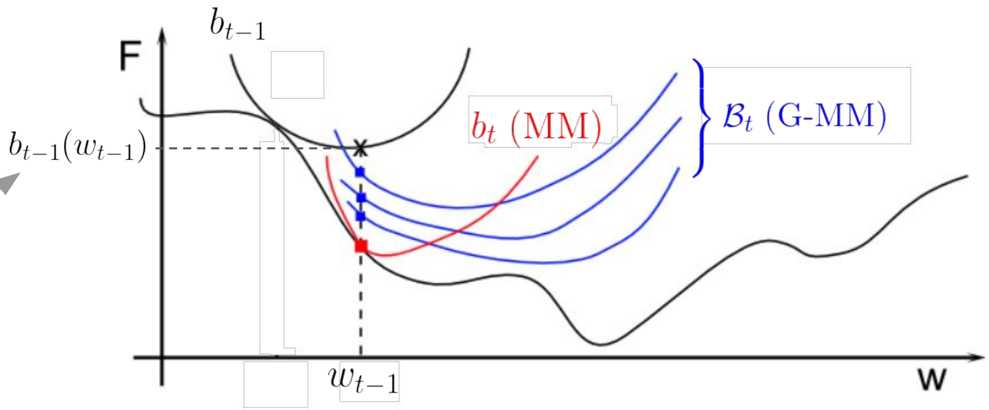

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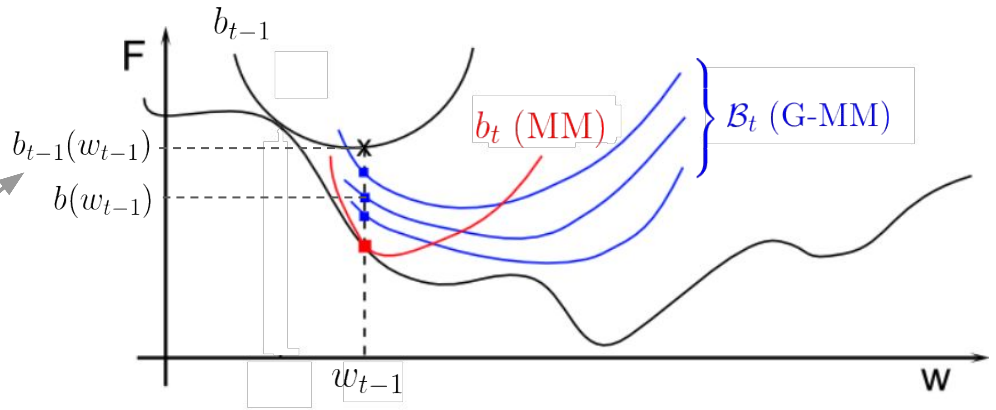

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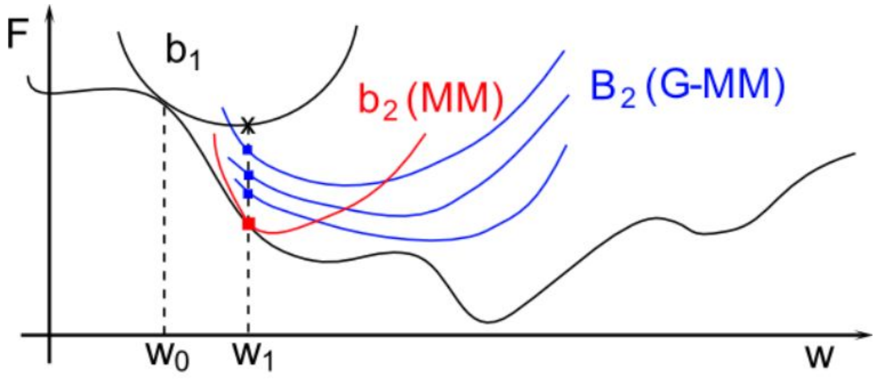
  - E.g. MM corresponds to $g(b, w_{t-1}) = -b(w_{t-1})$.

Generalized Majorization Minimization (G-MM)



G-MM constraint:

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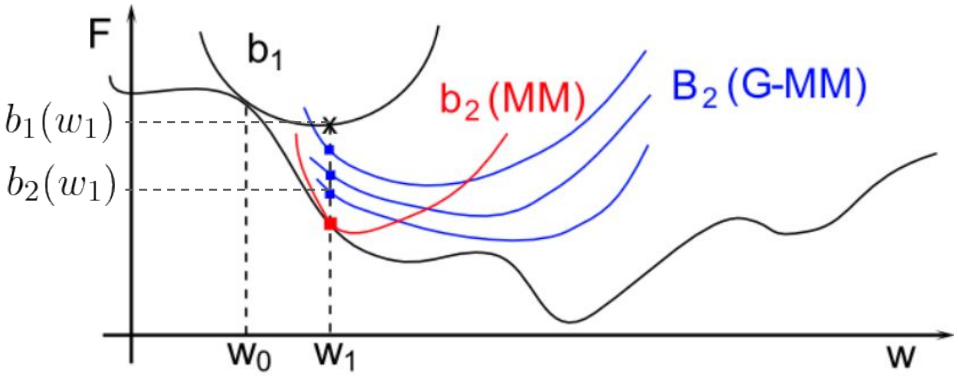

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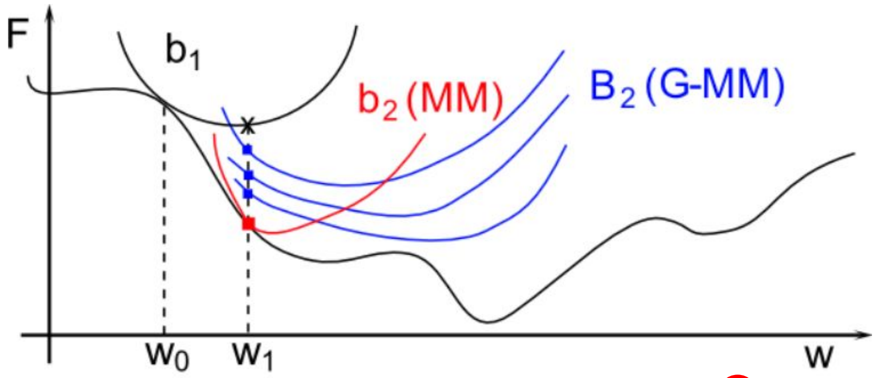

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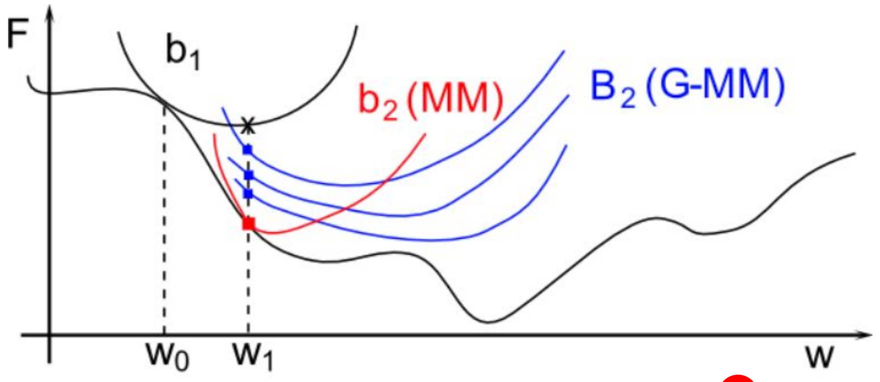

Theorem 1: $\lim_{T\to\infty} b_T(w_T) = F(w_T)$

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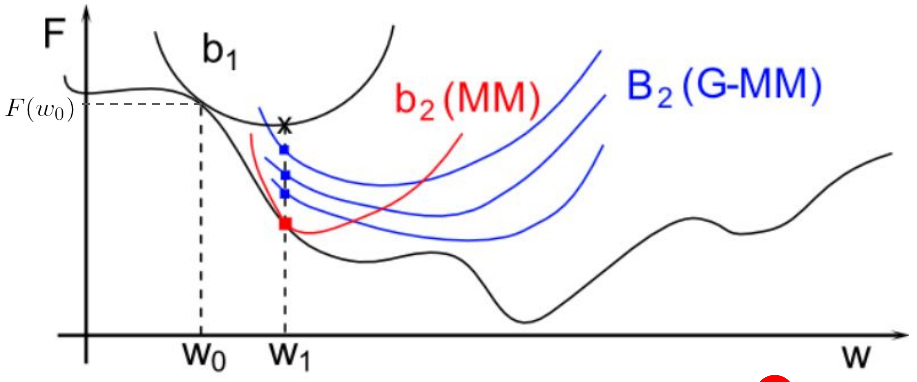

$F(w_0) \geq \{b_t(w_t)\}_{t=1}^{T}$ Non-increasing sequence

G-MM constraint:

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$$b_1(w_0) = F(w_0)$$

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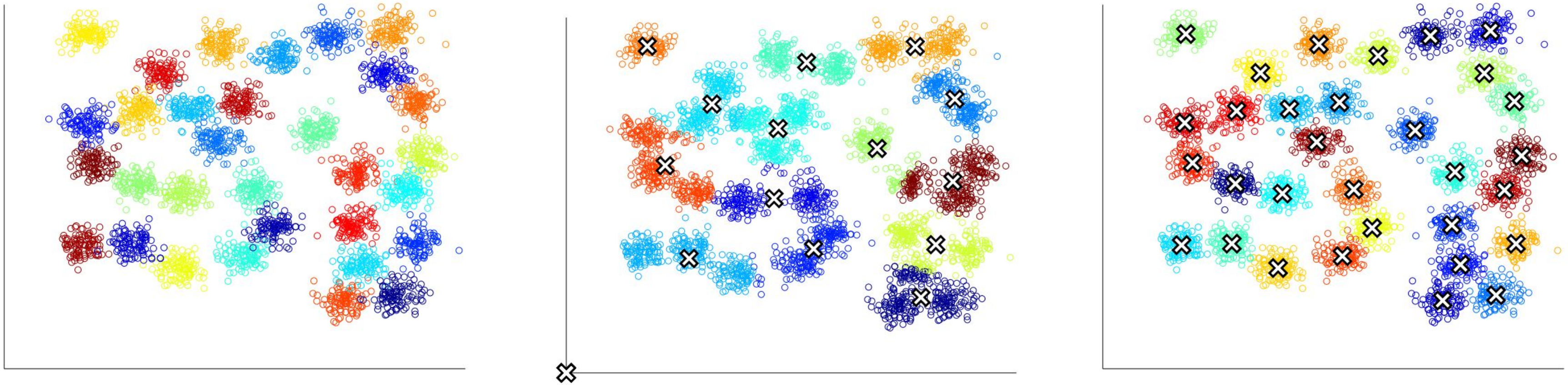
G-MM: Results on clustering



(a) ground truth (b) k-means (c) generalized-MM

Qualitative analysis of the solutions found by MM (figure b) and G-MM (figure c).

Summary

- We proposed G-MM, an iterative optimization framework that generalizes MM.
- MM requires bounds to ***touch*** the objective function, which leads to sensitivity to initialization.
- We show that this touching constraint is unnecessary and relax it in G-MM.
- MM measures progress w.r.t. objective values → $(F(w_t))_{t=0}^{T}$ is non-increasing.
- G-MM measures progress w.r.t. bound values → $(b_t(w_t))_{t=1}^{T}$ is non-increasing.

$$w_t = \arg\min_{w} b_t(w)$$

- In each iteration of G-MM, a new bound is chosen from a **set** of valid bounds $\mathcal{B}_t$.
- Our experimental results, on several non-convex optimization problems, show that …
  - G-MM is less sensitive to initialization.
  - G-MM converges to solutions that have better objective value and perform better on the task.
  - G-MM can inject randomness to the optimization framework by choosing $b_t \overset{U}{\sim} \mathcal{B}_t$.
  - G-MM can incorporate biases into the optimization framework by choosing $b_t = \underset{b \in \mathcal{B}_t}{\arg\max}\ g(b, w_{t-1})$.