Adaptive Regret of Convex and Smooth Functions

Lijun Zhang\textsuperscript{1}  Tie-Yan Liu\textsuperscript{2}  Zhi-Hua Zhou\textsuperscript{1}

\textsuperscript{1}National Key Laboratory for Novel Software Technology, Nanjing University

\textsuperscript{2}Microsoft Research Asia

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Online Learning

- Online Convex Optimization [Zinkevich, 2003]

1: for $t = 1, 2, \ldots, T$ do
2: Learner picks a decision $w_t \in W$
   Adversary chooses a function $f_t(\cdot) : W \mapsto \mathbb{R}$
3: Learner suffers loss $f_t(w_t)$ and updates $w_t$
4: end for

A classifier

$w_t \in \mathbb{R}^d$

An example $(x_t, y_t) \in \mathbb{R}^d \times \{\pm 1\}$
A loss $f_t(w) = \max(1 - y_t w^T x_t, 0)$

Cumulative Loss

$$\text{Cumulative Loss} = \sum_{t=1}^{T} f_t(w_t)$$

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Performance Measure

Regret

Regret = \sum_{t=1}^{T} f_t(w_t) - min_{w \in W} \sum_{t=1}^{T} f_t(w)

- Cumulative Loss of Online Learner
- Minimal Loss of Offline Learner
**Performance Measure**

- **Regret**

  \[
  \text{Regret} = \sum_{t=1}^{T} f_t(w_t) - \min_{w \in \mathcal{W}} \sum_{t=1}^{T} f_t(w)
  \]

  Cumulative Loss of Online Learner - Minimal Loss of Offline Learner

- **Convex Functions** [Zinkevich, 2003]
  - Online Gradient Descent (OGD)
    \[
    \text{Regret} = O\left(\sqrt{T}\right)
    \]

- **Convex and Smooth Functions** [Srebro et al., 2010]
  - OGD with prior knowledge
    \[
    \text{Regret} = O\left(1 + \sqrt{F_*}\right)
    \]
    where \(F_* = \min_{w \in \mathcal{W}} \sum_{t=1}^{T} f_t(w)\)

- **Exp-concave Functions** [Hazan et al., 2007]

- **Strongly Convex Functions** [Hazan et al., 2007]
Regret → Static Regret

$$\text{Regret} = \sum_{t=1}^{T} f_t(w_t) - \min_{w \in \mathcal{W}} \sum_{t=1}^{T} f_t(w)$$

$$= \sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(w_*)$$

where $w_* \in \arg\min_{w \in \mathcal{W}} \sum_{t=1}^{T} f_t(w)$

- $w_*$ is reasonably good during $T$ rounds

Changing Environments

- Different decisions will be good in different periods
- E.g., recommendation, stock market
Adaptive Regret

The Basic Idea

Minimize the regret over every interval $[r, s]$

$$
\text{Regret} \left([r, s]\right) = \sum_{t=r}^{s} f_t(w_t) - \min_{w \in \mathcal{W}} \sum_{t=r}^{s} f_t(w)
$$

- **Weakly Adaptive Regret** [Hazan and Seshadhri, 2007]
  
  $$
  \text{WA-Regret} (T) = \max_{[r, s] \subseteq [T]} \text{Regret} \left([r, s]\right)
  $$
  
  - The maximal regret over all intervals

- **Strongly Adaptive Regret** [Daniely et al., 2015]

  $$
  \text{SA-Regret} (T, \tau) = \max_{[s, s + \tau - 1] \subseteq [T]} \text{Regret} \left([s, s + \tau - 1]\right)
  $$

  - The maximal regret over all intervals of length $\tau$
Convex Functions [Jun et al., 2017]

\[
\text{Regret } ([r, s]) = O \left( \sqrt{(s - r) \log s} \right)
\]

\[
\Rightarrow \text{SA-Regret} (T, \tau) = O \left( \sqrt{\tau \log T} \right)
\]

Exp-concave Functions [Hazan and Seshadhri, 2007]

Strongly Convex Functions [Zhang et al., 2018]
Convex Functions [Jun et al., 2017]

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Exp-concave Functions [Hazan and Seshadhri, 2007]

Strongly Convex Functions [Zhang et al., 2018]

Question

Can smoothness be exploited to boost the adaptive regret?
Our Results

- Convex and Smooth Functions

\[ \text{Regret } ([r, s]) = O \left( \sqrt{\left( \sum_{t=r}^{s} f_t(w) \right) \log s \cdot \log(s - r)} \right) \]

- Become tighter when \( \sum_{t=r}^{s} f_t(w) \) is small

- Convex Functions [Jun et al., 2017]

\[ \text{Regret } ([r, s]) = O \left( \sqrt{(s - r) \log s} \right) \]
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- **Convex Functions** [Jun et al., 2017]

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  \text{Regret}([r, s]) = O \left( \sqrt{(s - r) \log s} \right)
  \]

- **Convex and Smooth Functions**

  \[
  \text{Regret}([r, s]) = O \left( \sqrt{\left( \sum_{t=r}^{s} f_t(w) \right) \log \sum_{t=1}^{s} f_t(w) \cdot \log \sum_{t=r}^{s} f_t(w)} \right)
  \]

  - Fully problem-dependent

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The Algorithm

- An Expert-algorithm
  - Scale-free online gradient descent [Orabona and Pál, 2018]
  - Can exploit smoothness automatically

- A Set of Intervals
  - Compact geometric covering intervals [Daniely et al., 2015]

\[
\begin{align*}
  t & \quad 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & \ldots \\
\end{align*}
\]

- A Meta-algorithm
  - AdaNormalHedge [Luo and Schapire, 2015]
  - Attain a small-loss regret and support sleeping experts

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Thanks!

Welcome to Our Poster @ Pacific Ballroom #161.

Strongly adaptive online learning.

Logarithmic regret algorithms for online convex optimization.

Adaptive algorithms for online decision problems.
Electronic Colloquium on Computational Complexity, 88.

Improved strongly adaptive online learning using coin betting.

Achieving all with no parameters: Adanormalhedge.
Scale-free online learning.

Smoothness, low-noise and fast rates.

Dynamic regret of strongly adaptive methods.
In *Proceedings of the 35th International Conference on Machine Learning*.

Online convex programming and generalized infinitesimal gradient ascent.