# Approximating Orthogonal Matrices with Effective Givens Factorization

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*joint work with*

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*Poster #164*

# Givens Factorization of Orthogonal Matrices

$$G^T(i,j,\alpha) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \cos(\alpha) & \cdots & -\sin(\alpha) & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \sin(\alpha) & \cdots & \cos(\alpha) & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

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## Exact Givens Factorization

$$U = G_1 \dots G_N \qquad N = \frac{d(d-1)}{2}$$

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1. Which orthogonal matrices can be effectively approximated?
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1. Which orthogonal matrices can be effectively approximated?
   (not all of them)
2. Which principles are behind effective approximation algorithms?
   (sparsity-inducing algorithms)

# Motivation: Unitary Basis Transform / FFT

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Once computed, applied many times

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Application: Graph Fourier Transform

# Which Matrices can be Effectively Approximated?

**Theorem**

*Let* $\epsilon > 0$*. If* $N = o\left(d^2 / \log(d)\right)$*, then as* $d \to \infty$*,*

$$\mu\left(\left\{U \in U(d) \middle| \inf_{G_1 \ldots G_N} \|U - \prod_n G_n\|_2 \leq \epsilon\right\}\right) \to 0 \ ,$$

*where* $\mu$ *is the Haar measure over* $U(d)$*.*

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*where* $\mu$ *is the Haar measure over* $U(d)$.

- proof is based on an $\epsilon$-covering argument
- suggests computational-to-statistical gap together with experimental results (details at poster)

## $K$-planted Distribution over $SO(d)$

**Sample** $U = G_1 \dots G_K$

- choose subspace $(i_k, j_k)$ uniformly with replacement
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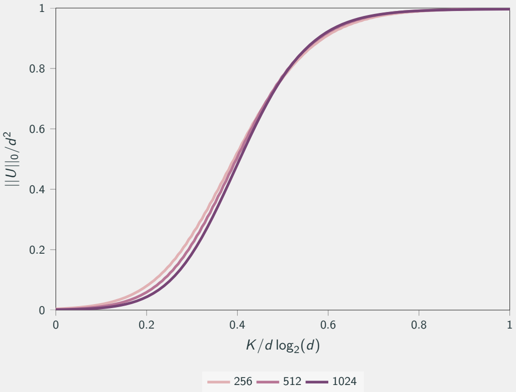


$K$-planted matrices quickly become dense

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- *Non-convex* greedy step
- global optimum in $\mathcal{O}(d^2)$ amortized time complexity

# Thank you

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`https://github.com/tfrerix/givens-factorization`