Multivariate Submodular Optimization

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Joint work with Bruce Shepherd (UBC)
Some Definitions

- Ground set $V = \{1, 2, \ldots, n\}$ with power set $2^V = \{A : A \subseteq V\}$
- A set function $f : 2^V \rightarrow \mathbb{R}$ is submodular if $\forall A \subseteq B$ and $v \notin B$:
  \[ f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B) \]
- Submodularity = diminishing returns property
- $f$ is monotone if $f(A) \leq f(B)$ for $A \subseteq B$
Submodularity in ML

- **Sensing & Information gathering:** Singh, Krause, Guestrin, Kaiser, Batalin ’07
- **Document summarization:** Lin and Bilmes ’11
- **Viral marketing:** Kempe, Kleinberg, Tardos ’03
- **Data subset selection & Active learning:** Wei, Iyer, Bilmes ’15
- **Robotics:** Dey, Liu, Herbert, Bagnell ’12
- **Feature selection:** Liu, Wei, Kirchhoff, Song, Bilmes ’13
- **Image segmentation:** Kim, Xing, Fei-Fei, Kanade ’11
- **Diversity:** Prasad, Jegelka, Batra ’14
Submodular Optimization

Given a submodular function $f$ and a family of feasible sets $\mathcal{F} \subseteq 2^V$:

**Submodular Optimization Problems:**

\[
\text{SO}(\mathcal{F}) \quad \min / \max f(S) : S \in \mathcal{F}
\]

where:

- $\mathcal{F} = \{S \subseteq V : |S| \leq k\}$
- $\mathcal{F} = \{S : S \subseteq V\}$
- $\mathcal{F} = \{$spanning trees of some graph $G$\}$
- $\mathcal{F} = \text{matroid or } p\text{-matroid intersection}$
Submodular Optimization Problems:

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Multi-Agent Submodular Optimization Problems:

\[ \text{MASO}(\mathcal{F}) \quad \min / \max \sum_{i=1}^{k} f_i(S_i) : S_1 \cup S_2 \cup \cdots \cup S_k \in \mathcal{F} \]

where \( \cup \) denotes the union of disjoint sets.
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**Multivariate Submodular Optimization Problems:**

$$\text{MVSO}(\mathcal{F}) \quad \min / \max g(S_1, \ldots, S_k) : S_1 \uplus S_2 \uplus \cdots \uplus S_k \in \mathcal{F}$$
Our Results

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\text{SO}(\mathcal{F}) \quad \min / \max f(S) : S \in \mathcal{F}
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**Question 1:** Is MVSO really more general than MASO?
Our Results

SO(\mathcal{F}) \quad \min / \max f(S) : S \in \mathcal{F}

MASO(\mathcal{F}) \quad \min / \max \sum_{i=1}^{k} f_i(S_i) : S_1 \cup S_2 \cup \cdots \cup S_k \in \mathcal{F}

MVSO(\mathcal{F}) \quad \min / \max g(S_1, \ldots, S_k) : S_1 \cup S_2 \cup \cdots \cup S_k \in \mathcal{F}

Question 1: Is MVSO really more general than MASO? Yes!

(MV – Min) \quad \min_{\text{s.t.}} g(S_1, \ldots, S_k) : S_1 \cup \cdots \cup S_k = V

(MA – Min) \quad \min_{\text{s.t.}} \sum_{i=1}^{k} f_i(S_i) : S_1 \cup \cdots \cup S_k = V

Theorem

There is a tight $\tilde{\Omega}(n)$ gap between the approximation factors for MV-Min and MA-Min where all the functions are nonnegative and monotone.
Our Results

\[
\begin{align*}
\text{SO}(\mathcal{F}) & \quad \min / \max f(S) : S \in \mathcal{F} \\
\text{MASO}(\mathcal{F}) & \quad \min / \max \sum_{i=1}^{k} f_i(S_i) : S_1 \cup S_2 \cup \cdots \cup S_k \in \mathcal{F} \\
\text{MVSO}(\mathcal{F}) & \quad \min / \max g(S_1, \ldots, S_k) : S_1 \cup S_2 \cup \cdots \cup S_k \in \mathcal{F}
\end{align*}
\]

Question 2: Given an \(\alpha\)-approx for \(\text{SO}(\mathcal{F})\), what can be said about \(\text{MVSO}(\mathcal{F})\)? [We refer to the additional incurred loss as the \text{MV gap}]

Theorem (Maximization)

The MV gap of \(1 - 1/e\) for monotone functions and 0.385 for nonmonotone

Theorem (Minimization)

Essentially tight approximation factors w.r.t. the curvature of \(g\)
Our Results

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\text{SO}(\mathcal{F}) \quad \min / \max f(S) : S \in \mathcal{F}
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\text{MVSO}(\mathcal{F}) \quad \min / \max g(S_1, \ldots, S_k) : S_1 \cup S_2 \cup \cdots \cup S_k \in \mathcal{F}
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Question 2: Given an \(\alpha\)-approx for \(\text{SO}(\mathcal{F})\), what can be said about \(\text{MVSO}(\mathcal{F})\)? [We refer to the additional incurred loss as the MV gap]

**Theorem (Maximization)**

- MV gap of \(1 - 1/e\) for monotone functions and 0.385 for nonmonotone
- MV gap of 1 for several families such as matroids and \(p\)-systems
- Accelerated greedy and distributed algorithms still work for MVSO

**Theorem (Minimization)**

- Essentially tight approximation factors w.r.t. the curvature of \(g\)