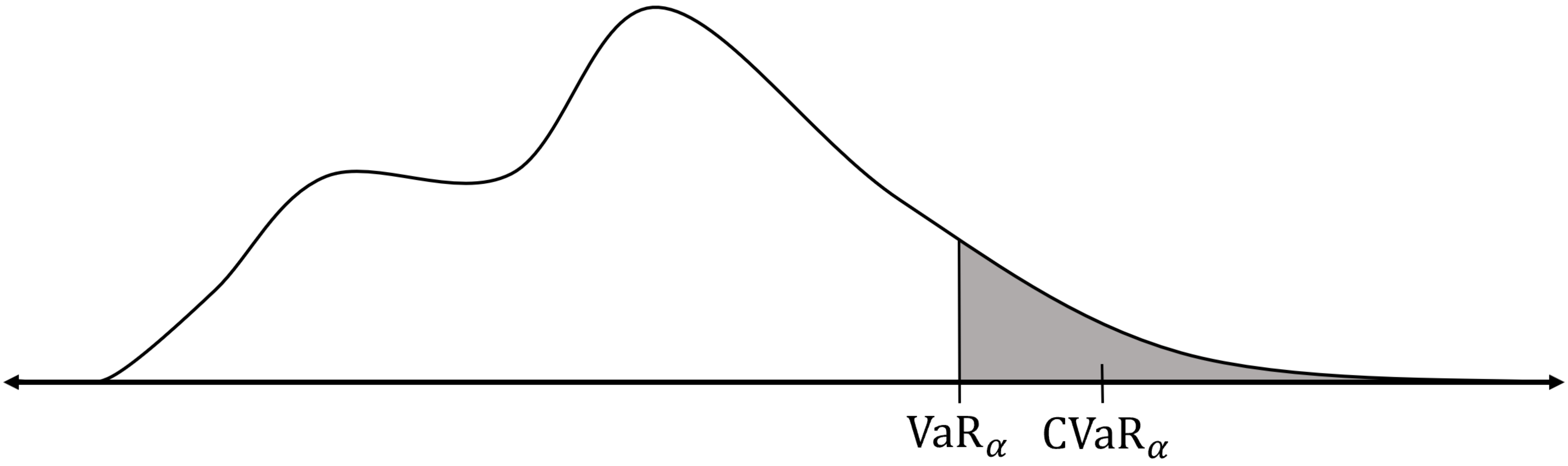
Concentration Inequalities for Conditional Value at Risk

Philip S. Thomas and Erik Learned-Miller

College of Information and Computer Sciences

University of Massachusetts Amherst

Conditional Value at Risk
Expected Shortfall



Concentration Inequality for CVaR (Brown 2007)

Theorem 1. *If* $\text{supp}(X) \subseteq [a, b]$ *and* X *has a continuous distribution function, then for any* $\delta \in (0, 1]$,

$$\Pr\left(\mathrm{C}(X) \leq \widehat{\mathrm{C}} + (b-a)\sqrt{\frac{5\ln(3/\delta)}{\alpha n}}\right) \geq 1 - \delta.$$

Theorem 2. *If* $\text{supp}(X) \subseteq [a, b]$, *then for any* $\delta \in (0, 1]$,

$$\Pr\left(\mathrm{C}(X) \geq \widehat{\mathrm{C}} - \frac{b-a}{\alpha}\sqrt{\frac{\ln(1/\delta)}{2n}}\right) \geq 1 - \delta.$$

New Concentration Inequalities for CVaR

Theorem 3. *If $X_1, \ldots, X_n$ are independent and identically distributed random variables and $\Pr(X_1 \leq b) = 1$ for some finite b, then for any $\delta \in (0, 0.5]$,*

$$\Pr\left(\text{CVaR}_\alpha(X_1) \leq Z_{n+1} - \frac{1}{\alpha}\sum_{i=1}^{n}(Z_{i+1} - Z_i)\left(\frac{i}{n} - \sqrt{\frac{\ln(1/\delta)}{2n}} - (1-\alpha)\right)^+\right) \geq 1 - \delta,$$

where $Z_1, \ldots, Z_n$ are the order statistics (i.e., $X_1, \ldots, X_n$ sorted in ascending order), $Z_{n+1} = b$, and $x^+ := \max\{0, x\}$ for all $x \in \mathbb{R}$.

Theorem 4. *If $X_1, \ldots, X_n$ are independent and identically distributed random variables and $\Pr(X_1 \geq a) = 1$ for some finite a, then for any $\delta \in (0, 0.5]$,*

$$\Pr\left(\text{CVaR}_\alpha(X_1) \geq Z_n - \frac{1}{\alpha}\sum_{i=0}^{n-1}(Z_{i+1} - Z_i)\left(\min\left\{1, \frac{i}{n} + \sqrt{\frac{\ln(1/\delta)}{2n}}\right\} - (1-\alpha)\right)^+\right) \geq 1 - \delta,$$

where $Z_1, \ldots, Z_n$ are the order statistics (i.e., $X_1, \ldots, X_n$ sorted in ascending order), $Z_0 = a$, and where $x^+ := \max\{0, x\}$ for all $x \in \mathbb{R}$.

Figure 2. Main results, presented in a standalone fashion, and where $x^+ := \max\{0, x\}$ for all $x \in \mathbb{R}$.

Concentration Inequalities for CVaR

- **Theorem 5**: Our lower bound is a *strict* improvement on Brown’s if $n > 3$.
- **Empirical Results**: Our bounds are significant improvements on Brown’s.
- **Drawbacks**:
  - Our inequalities are not in as neat of a form
  - Our inequalities have the same asymptotic rates as Brown’s

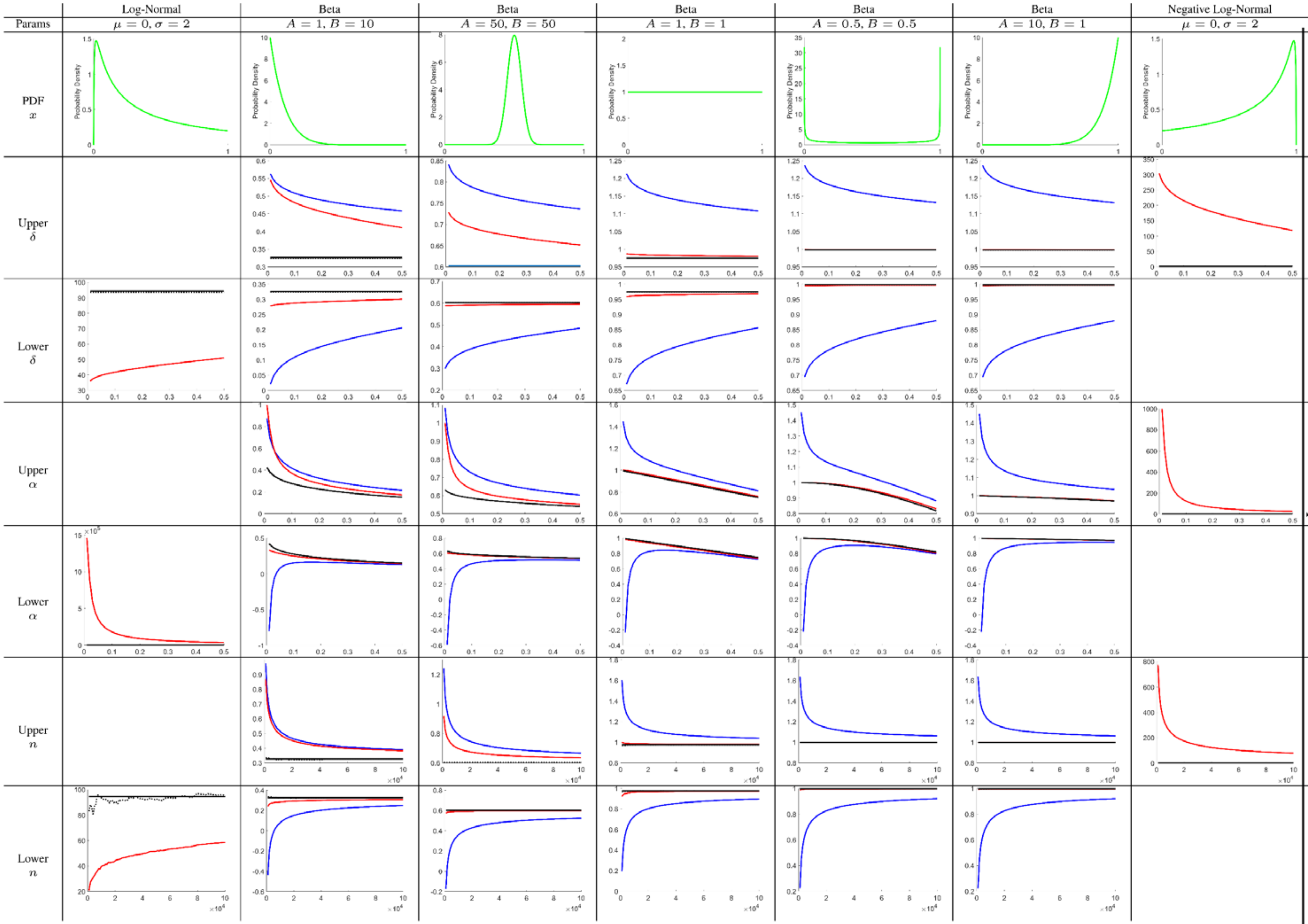


Figure 5. In all cases, red = Theorems 3 and 4, blue = Brown, dotted-black = sample CVaR, and solid-black = actual CVaR. On the left, the second line is the horizontal axis label. In some cases the actual CVaR (solid-black) obscures the sample CVaR (dotted-black) and our theorems (red).

Come chat at the poster this evening!