Inference and Sampling of $K_{33}$-free Ising Models

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Definitions and Notations

For a graph $G = (V, E)$, $|V| = N$, zero-field Ising model is a distribution over $S \in \{-1, +1\}^N$ defined as

$$
\mathbb{P}(S = X) = \frac{1}{Z} \exp\left( \sum_{e=\{v,w\} \in E} J_e x_v x_w \right)
$$

(1)

where $\{J_e\}_{e \in E}$ are pairwise interactions and

$$
Z(J) = \sum_{X \in \{-1,+1\}^N} \exp\left( \sum_{e=\{v,w\} \in E} J_e x_v x_w \right)
$$

(2)

is a partition function.
Problem Overview

Question
For which graphs $G$ can we compute $Z$ and sample from $\mathbb{P}(S)$?

Fact (Barahona, 1982)
Even when $G$ is a two-level square grid, the task of finding $Z$ is NP-hard.

Fact (Jerrum & Sinclair, 1993)
Even when $J > 0$, the task of finding $Z$ is $\#P$-complete.
Problem Overview: Planar Zero-field Ising Models

Planar zero-field Ising model - a case when $G$ is planar.

Theorem

Given a planar zero-field Ising model, finding $Z$ and sampling from $\mathbb{P}(S)$ takes $O(N^{3/2})$ time.

- Theorem is due to (Kasteleyn, 1963; Wilson, 1997; Schraudolph & Kamenetsky, 2009; Thomas & Middleton, 2009; 2013).
- No self-contained description of the algorithm.
- Extension to arbitrary genus $g$ with a factor of $4^g$ (Gallucio & Loebl, 1999).
Algorithm Overview: Graph Decomposition

Informal definition

A tree of triconnected components $T$ of graph $G$ is a tree decomposition of $G$ into triconnected graphs $G_t$ with shared edges.

Theorem (Hopcroft & Tarjan, 1973)

A tree of triconnected components is unique and can be obtained in $O(N + |E|)$. 
Algorithm Overview: Inference of $K_{33}$-free Zero-field Ising Models

**Lemma (Hall, 1943)**

*Graph $G$ is $K_{33}$-free if and only if its triconnected components are either planar or $K_5$.***

**Theorem**

*Given a $K_{33}$-free zero-field Ising model, finding $Z$ and sampling from $P(S)$ takes $O(N^{3/2})$ time.*
Conclusions

Main results:

- Self-contained description of $O(N^{3/2})$ inference and sampling of planar zero-field Ising models.
- $O(N^{3/2})$ inference and sampling of $K_{33}$-free Ising models.

Poster: “Inference and Sampling of $K_{33}$-free Ising Models”, Valerii Likhosherstov, Yury Maximov, Michael Chertkov.

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