Random Walks on Hypergraphs with Edge-Dependent Vertex Weights

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Graphs in Machine Learning

Graphs model pairwise relationships between objects

Examples:
- Social networks
- Internet
- Biological systems
Graphs in Machine Learning

However, graphs may lose information about the relationships between objects.
Graphs in Machine Learning

However, graphs may lose information about the relationships between objects.

Example: Given a co-authorship network, which authors wrote which papers?

Above: a fictitious network of authors.
Graphs in Machine Learning

However, graphs may lose information about the relationships between objects.

Example: Given a co-authorship network, which authors wrote which papers?

Above: a fictitious network of authors.
A hypergraph $H = (V, E)$ models higher order relationships.

$E \subseteq 2^V$ is a set of hyperedges. Each hyperedge $e \in E$ can contain $> 2$ vertices.
Graphs

Model pairwise relationships

Hypergraphs

Model higher-order relationships
Hypergraphs in Machine Learning

Zhou, Huang, and Schölkopf [NeurIPS 2006]:

• Adapt spectral clustering methods to hypergraphs by defining a hypergraph Laplacian matrix

• demonstrate improvements over graphs in classification tasks
Do Hypergraphs Model Higher-Order Information?

However, Agarwal et al. [ICML 2006] show that Zhou et al. are really doing inference on graphs.
Specifically, Agarwal et al. shows that Zhou et al.’s hypergraph Laplacian matrix (and others in the literature) are equal to Laplacians of: either **clique** graph, or **star** graph.
Do Hypergraphs Model Higher-Order Information?

**Question:** When do hypergraph learning algorithms not reduce to graph algorithms?
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Our work: When the hypergraph has **edge-dependent** vertex weights.
What are Edge-Dependent Vertex Weights?

A vertex $v$ has weight $\gamma_e(v)$ for each incident hyperedge $e$.

$\gamma_e(v)$ describes the contribution of vertex $v$ to hyperedge $e$.

Example: in co-authorship network, edge-dependent vertex weights can measure the contribution of each author to a paper.

\[ \begin{align*}
\gamma_e(a) &= 5 \\
\gamma_e(b) &= 3 \\
\gamma_e(c) &= 1
\end{align*} \]
In contrast, edge-independent vertex weights: $\gamma_e(v) = \gamma_f(v)$ for all hyperedges $e, f$ incident to $v$

Most hypergraph literature assumes edge-independent vertex weights. (Typically the vertex weights are 1.)
Part 1: Edge-Independent Vertex Weights

We show: When vertex weights are edge-independent, then random walks on hypergraph = random walks on clique graph

(Formally, the random walks have equal probability transition matrices)
Part 1: Edge-Independent Vertex Weights

Thus, existing hypergraph Laplacian matrices (e.g. Zhou et al.) are equal to Laplacian matrix of a clique graph

This is because these Laplacians are derived from random walks on hypergraphs with **edge-independent** vertex weights

\[
\begin{align*}
\gamma(a) &= 2 \\
\gamma(b) &= 1 \\
\gamma(c) &= 1 \\
\gamma(d) &= 2
\end{align*}
\]
Part 1: Edge-Independent Vertex Weights

Thus, existing hypergraph Laplacian matrices (e.g. Zhou et al.) are equal to Laplacian matrix of a clique graph

This is because these Laplacians are derived from random walks on hypergraphs with edge-independent vertex weights.

Generalizing Agarwal et al, we give the underlying reason that hypergraphs with edge-independent vertex weights do not utilize higher-order relations between objects.
Conversely, we show that random walks on hypergraphs with edge-dependent vertex weights ≠ random walks on clique graph.

Formally, there exists such a hypergraph whose random walk is not the same as a random walk on clique graph for any choice of edge weights.

Part 2: Edge-Dependent Vertex Weights

\[ \begin{align*}
\gamma_{e_1}(a) &= 2 \\
\gamma_{e_1}(b) &= 1 \\
\gamma_{e_1}(c) &= 1 \\
\gamma_{e_2}(a) &= 1 \\
\gamma_{e_2}(c) &= 1 \\
\gamma_{e_2}(d) &= 1
\end{align*} \]
Part 2: Edge-Dependent Vertex Weights

Conversely, we show that random walks on hypergraphs with edge-dependent vertex weights ≠ random walks on clique graph. Formally, there exists such a hypergraph whose random walk is not the same as a random walk on clique graph for any choice of edge weights.

Thus, hypergraphs with edge-dependent vertex weights utilize higher-order relations between objects.
Motivated by this result, we develop a **spectral theory** for hypergraphs with edge-dependent vertex weights.

### Part 3: Theory for Edge-Dependent Vertex Weights

<table>
<thead>
<tr>
<th></th>
<th>Graphs</th>
<th>Hypergraphs with edge-dependent vertex weights</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stationary distribution</strong></td>
<td>$\pi_v = \rho \sum_{e \in E(v)} w(e)$</td>
<td>$\pi_v = \sum_{e \in E(v)} \rho_e \omega(e) \gamma_e(v)$</td>
</tr>
<tr>
<td><strong>Mixing time of random walk</strong></td>
<td>$t_{mix}^G(\epsilon) = \frac{2}{\Phi^2} \log \left( \frac{1}{2\epsilon \sqrt{d_{\text{min}}}} \right)$</td>
<td>$t_{mix}^H(\epsilon) = \frac{8\beta_1}{\Phi^2} \log \left( \frac{1}{2\epsilon \sqrt{d_{\text{min}} \beta_2}} \right)$</td>
</tr>
<tr>
<td><strong>Laplacian matrix + Cheeger inequality</strong></td>
<td>$L = D - A$</td>
<td>$L = \Pi - \frac{\Pi P + P^T \Pi}{2}$</td>
</tr>
</tbody>
</table>
Part 4: Experiments

We demonstrate two applications of edge-dependent vertex weights:

1. Ranking authors in citation network
2. Ranking players in a **multiplayer** video game

\[
y_e(v) = \begin{cases} 
2 & \text{if vertex } v \text{ is the first or last author of paper,} \\
1 & \text{if vertex } v \text{ is a middle author of paper.}
\end{cases}
\]
Thank you for listening!

Check out our poster: #216 at the Pacific Ballroom, tonight at 6:30 – 9pm
Our full paper is also in ICML 2019 proceedings and on arXiv.

Random Walks on Hypergraphs with Edge-Dependent Vertex Weights
Uthsav Chitra, Benjamin J Raphael
(Submitted on 20 May 2019)

Hypergraphs are used in machine learning to model higher-order relationships in data. While spectral methods for graphs are well-established, spectral theory for hypergraphs remains an active area of research. In this paper, we use random walks to develop a spectral theory for hypergraphs with edge-dependent vertex weights: hypergraphs where every vertex $v$ has a weight $\gamma_e(v)$ for each incident hyperedge $e$ that describes the contribution of $v$ to the hyperedge $e$. We derive a random walk-based hypergraph Laplacian, and bound the mixing time of random walks on such hypergraphs. Moreover, we give conditions under which random walks on such hypergraphs are equivalent to random walks on graphs. As a corollary, we show that current machine learning methods that rely on Laplacians derived from random walks on hypergraphs with edge-independent vertex weights do not utilize higher-order relationships in the data. Finally, we demonstrate the advantages of hypergraphs with edge-dependent vertex weights on ranking applications using real-world datasets.