Returning the Favour:
When Regression Benefits from Probabilistic Causal Knowledge

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* Equal contribution

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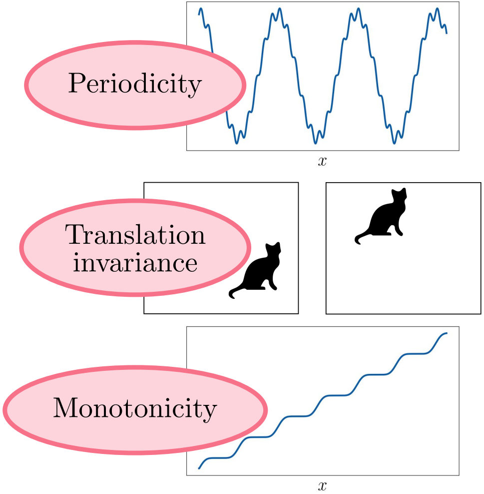
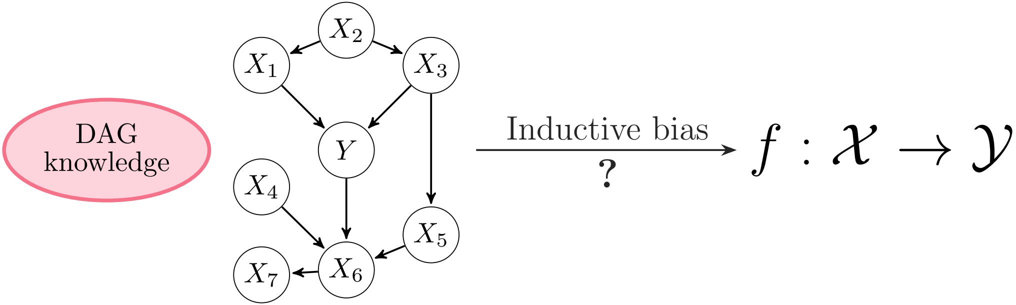
Central Question

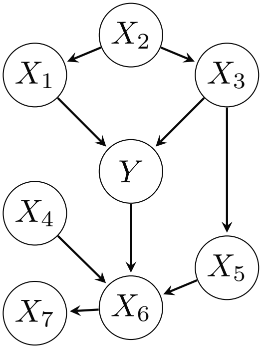


$$\xrightarrow{\text{Inductive bias}} f : \mathcal{X} \to \mathcal{Y}$$

Central Question



What does a DAG tell us about $\mathbb{P}(Y|X)$?



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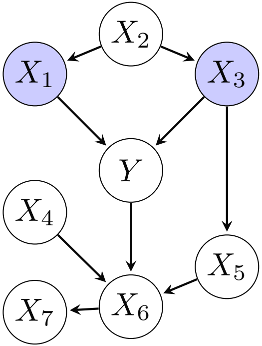

Parents of Y

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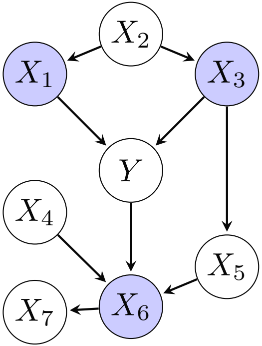

Parents of Y $\cup$ Children of Y

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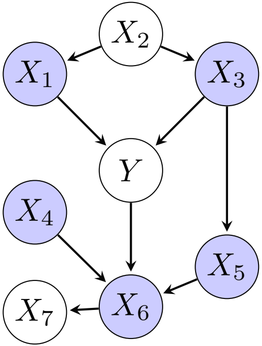

Parents of Y $\cup$ Children of Y $\cup$ Spouses of Y

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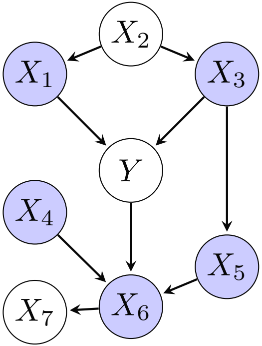

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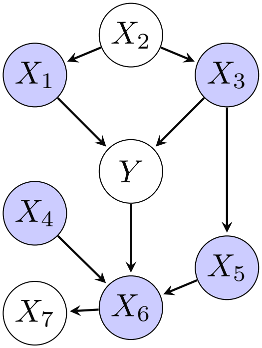

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- The Markov Boundary contains all relevant information for $\mathbb{P}(Y|X)$

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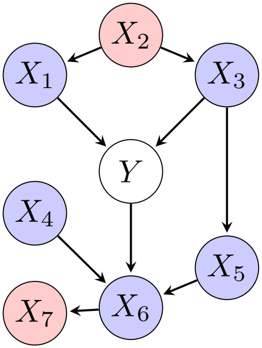

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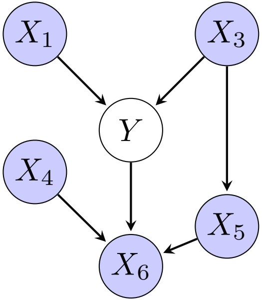

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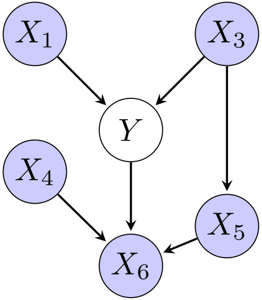
Further information in the Markov Boundary?



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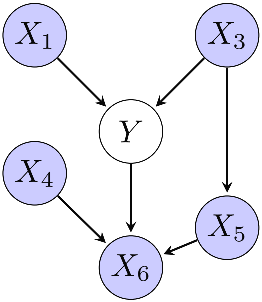

- Presence of **collider** X_6

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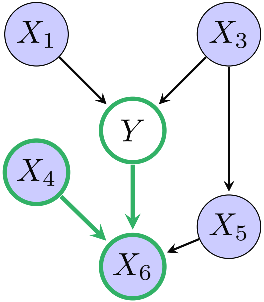

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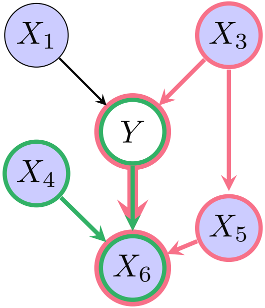

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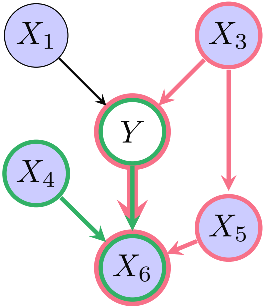
$$Y \perp\!\!\!\perp X_4$$

$$Y \perp\!\!\!\perp X_5 \mid X_3$$



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- Colliders provide additional information on $\mathbb{P}(Y|X)$ which is unused

Inductive biases from simple collider

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$$Y \rightarrow X_1 \leftarrow X_2$$

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Squared-loss regression problem

$$\boxed{\operatorname*{arg\,min}_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \left(y_i - f(x_{1i}, x_{2i})\right)^2 + \lambda \Omega(f)}$$

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$$\mathbb{E}[f^*(X_1, X_2)|X_2]$$

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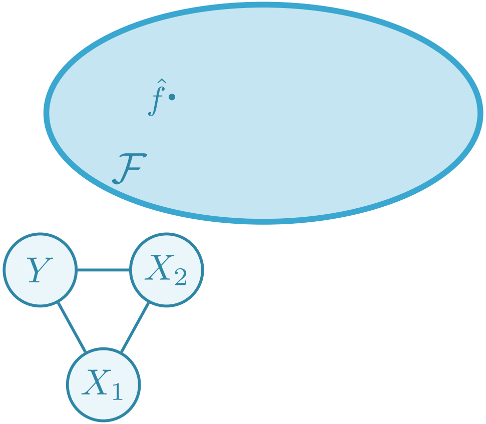
Objective

Find a regressor $\hat{f} : \mathcal{X} \rightarrow \mathcal{Y}$ in hypothesis space $\mathcal{F}$ that satisfies

$$
\boxed{\hat{f} \in \{f \in \mathcal{F} \mid \mathbb{E}[f(X_1, X_2) \mid X_2] = 0\}\,.}
$$

Constraining the hypothesis space $\mathcal{F}$



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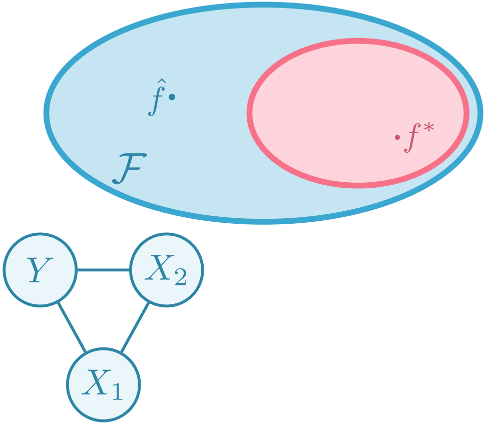

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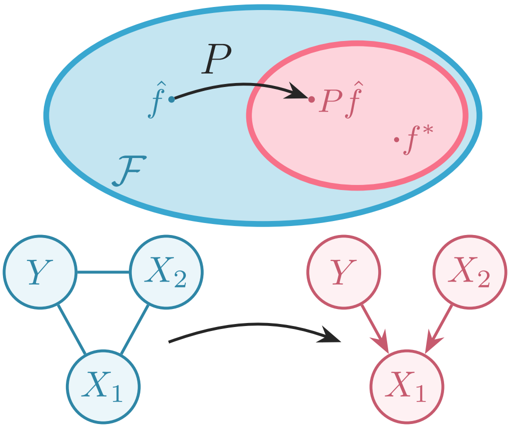
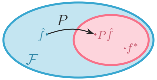


$$\left\{f \in \mathcal{F} \mid \mathbb{E}[f(X_1, X_2) \mid X_2] = 0\right\} = \mathrm{Range}(P)$$

Collider regression in $\mathcal{F} = L^2(X)$



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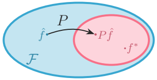

Take $P : L^2(X) \to L^2(X)$,

$$Pf(x_1, x_2) = f(x_1, x_2) - \mathbb{E}[f(X_1, X_2) | X_2 = x_2].$$

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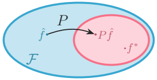

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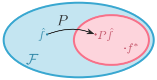

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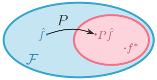

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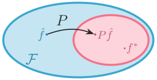
Proposition

Let $h \in L^2(X)$, then we have

$$\Delta(h, Ph) = \mathbb{E}[(Y - h(X))^2] - \mathbb{E}[(Y - Ph(X))^2] = \|(\mathrm{Id} - P)h\|^2_{L^2(X)}.$$

Collider regression in a RKHS $\mathcal{F} = \mathcal{H}$



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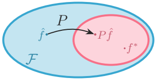

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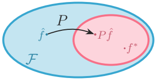

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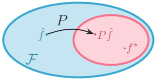

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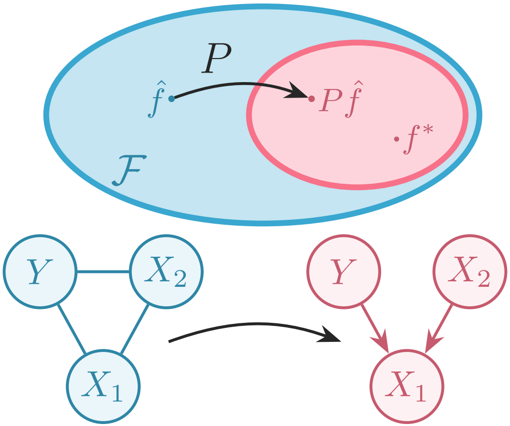
Theorem

Assume $M = \sup_{x \in \mathcal{X}} k(x, x) < \infty$ and $\mathrm{Var}(Y|X) \geq \eta > 0$. Then, the generalisation gap between $\hat{f}$ and $P\hat{f}$ satisfies

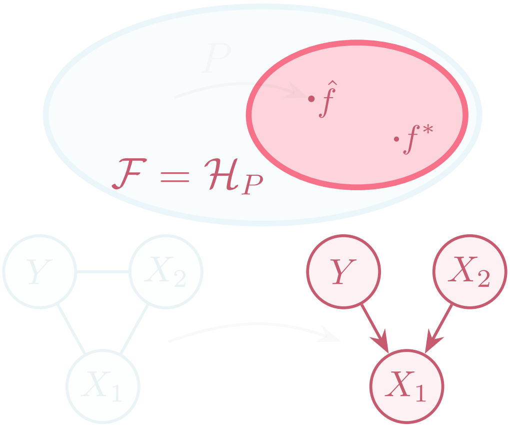
$$\mathbb{E}[\Delta(\hat{f}, P\hat{f})] \geq \frac{\eta \mathbb{E}\left[\|\mu_{X|X_2}(X)\|^2_{L^2(X)}\right]}{\left(\sqrt{n}M + \lambda/\sqrt{n}\right)^2} \tag{1}$$

where $\mu_{X|X_2} = \mathbb{E}[k(X, \cdot)|X_2]$ is a RKHS representation of $\mathbb{P}(X|X_2)$.



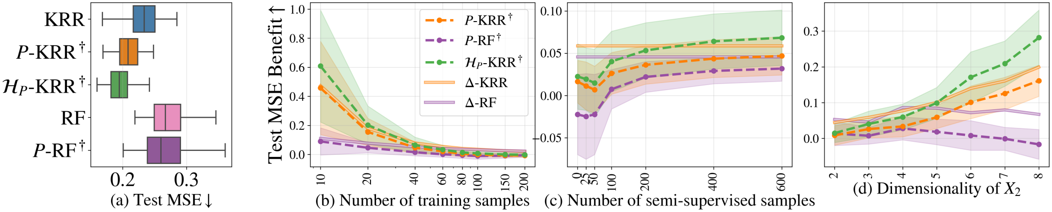
$\mathcal{F} = \mathcal{H}$ with kernel $k(x, x') = \langle k(x, \cdot), k(x', \cdot) \rangle$

Collider regression in a RKHS $\cancel{\mathcal{F} = \mathcal{H}}$ $\mathcal{F} = \mathcal{H}_P$



$\mathcal{F} = \mathcal{H}_P$ with kernel $k_P(x, x') = \langle P^* k(x, \cdot), P^* k(x', \cdot) \rangle$

Simulation experiment



(a) : Test MSEs for the simulation experiment ; dataset is generated using $d_1 = 3$, $d_2 = 3$, $n = 50$ and 100 semi-supervised samples ; experiments is run for 100 datasets generated with different seeds ; statistical significance is confirmed via Wilcoxon signed-rank ; (b, c, d) : Ablation study on the number of training samples, number of semi-supervised samples and dimensionality of X_2.

Table 1: MSE, signal-to-noise ratio (SNR) and correlation on test data for the aerosol radiative forcing experiment ; $n = 50$ and 200 semi-supervised samples ; statistical significance is confirmed via Wilcoxon signed-rank ; experiments are run for 100 datasets generated by FaIR with different seeds ; $\uparrow/\downarrow$ indicates higher/lower is better; we report 1 standard deviation; † indicates our proposed methods.

	MSE $\downarrow$	SNR $\uparrow$	Correlation $\uparrow$
RF	$0.90_{\pm 0.04}$	$0.44_{\pm 0.19}$	$0.32_{\pm 0.08}$
P-RF†	$\mathbf{0.89}_{\pm \mathbf{0.03}}$	$\mathbf{0.49}_{\pm \mathbf{0.15}}$	$\mathbf{0.34}_{\pm \mathbf{0.07}}$
KRR	$0.88_{\pm 0.04}$	$0.56_{\pm 0.21}$	$0.37_{\pm 0.05}$
P-KRR†	$\mathbf{0.86}_{\pm \mathbf{0.03}}$	$\mathbf{0.65}_{\pm \mathbf{0.14}}$	$\mathbf{0.40}_{\pm \mathbf{0.02}}$
$\mathcal{H}_P$-KRR†	$\mathbf{0.86}_{\pm \mathbf{0.03}}$	$\mathbf{0.64}_{\pm \mathbf{0.14}}$	$0.39_{\pm 0.03}$

Outlooks

- Collider structures within causal graphs consitute a useful form of inductive bias within supervised learning

To find out more come to our poster:
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Outlooks

- Collider structures within causal graphs consitute a useful form of inductive bias within supervised learning
- Provable benefit of semi-supervised learning arising from causal structure
- Collider regression for more general DAGs in the paper
- Similar reasoning can be applied to other forms of inference about $\mathbb{P}(Y|X)$, e.g. classification or quantile regression

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