# The Computational Complexity of Concise Hypersphere Classification

Iyad Kanj

School of Computing, DePaul University

Joint work with: Eduard Eiben (Royal Holloway University of London) & Robert Ganian (TU-Vienna) & Sebastian Ordyniak (University of Leeds) & Stefan Szeider (TU-Vienna)

# Problem Definition

**Binary Hypersphere Classification (BHC)**
**Input:** A set $V = V_R \cup V_B$ of $d$-dimensional vectors over $D = \{0, 1\}$ where $V_R \cap V_B = \emptyset$.
**Question:** Is there a vector $\vec{c} \in D^d$ and $r \in \mathbb{N}$ such that $V_B \subseteq B(\vec{c}, r)$ and $V_R \cap B(\vec{c}, r) = \emptyset$?

# Motivation

- A classical classification problem
- Hypersphere separation is simple to explain (Explainable AI)
- Important applications in machine learning
- Extensively studied by the ML and computational geometry communities

# Goal

**Aim**: understand the fine-grained complexity of **BHC**

- When can **BHC** be solved efficiently?
- Design exact algorithms with runtime guarantees that exploit the structure of the input
- Understand the impacts of **conciseness** on the problem complexity

# Tool: Parameterized Complexity

**Parameterized Complexity** refines Classical Complexity in order to account for the presence of a numerical parameter(s) $\kappa$ which characterizes some property of the input $I$

It is a well-established and fundamental framework, originally introduced in the setting of discrete graph algorithms

# Tool: Parameterized Complexity

Basic question: Can an NP-hard problem $\mathcal{P}$ be solved more efficiently when the parameter(s) is small?

1. **No**—$\mathcal{P}$ remains NP-hard even for fixed values of $\kappa$
2. **Yes**—$\mathcal{P}$ parameterized by $\kappa$ is in the class FPT
   - $\mathcal{P}$ can be solved by an algorithm with runtime $f(\kappa) \cdot |I|^{\mathcal{O}(1)}$ for some function $f$
   - Polynomial-time for every fixed value of $\kappa$, with fixed polynomial factor
3. **A little**—$\mathcal{P}$ parameterized by $\kappa$ is W[1]-hard (and in the class XP)
   - $\mathcal{P}$ can be solved by an algorithm with runtime $|I|^{f(\kappa)}$ for some function $f$
   - Polynomial-time for every fixed value of $\kappa$, but bad scaling

# Parameters Under Consideration

We study the complexity of BHC w.r.t. the following:

1. The cardinalities of $V_R$ and $V_B$
2. The treewidth of the incidence graph
3. The data conciseness (maximum number of 1's per red/blue vector) and the explanation conciseness (maximum number of 1's in the sought hypersphere center)

# Results

1. We show that BHC remains NP-complete in severely restricted settings: when there are only two red vectors or only two blue vectors

2. We show that BHC is FPT parameterized by the number of red vectors plus the number of blue vectors, and hence, by the above, this parameterization is in fact tight: one cannot drop any of the two parameters without losing tractability

# Results

3. We show that BHC is XP parameterized by the treewidth of the incidence graph
4. For conciseness, we show:
   - BHC can be solved in polynomial time if the data conciseness is at mot 3 and becomes NP-hard if it is at least 4
   - BHC is W[2]-hard parameterized by the explanation conciseness

# Results: Summary Table

| Structure | $\emptyset$ | *econ* | *dcon* | *econ* + *dcon* |
|---|---|---|---|---|
| $\emptyset$ | NP-h | XP, W[2]-h | NP-h$_{\geq 4}$ | FPT |
| $\lvert V_R \rvert$ | NP-h$_{\geq 2}$ | XP, W[2]-h | FPT | FPT |
| $\lvert V_B \rvert$ | NP-h$_{\geq 2}$ | XP, W[1]-h | FPT | FPT |
| $\lvert V_R \rvert + \lvert V_B \rvert$ | FPT | FPT | FPT | FPT |
| $d$ | FPT | FPT | FPT | FPT |
| $tw$ | XP | FPT | FPT | FPT |