Tight and Robust Private Mean Estimation with Few Users

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Outline

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1) Introduction

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2) Results

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3) Overview of Algorithm

Establishing Privacy of Released Data

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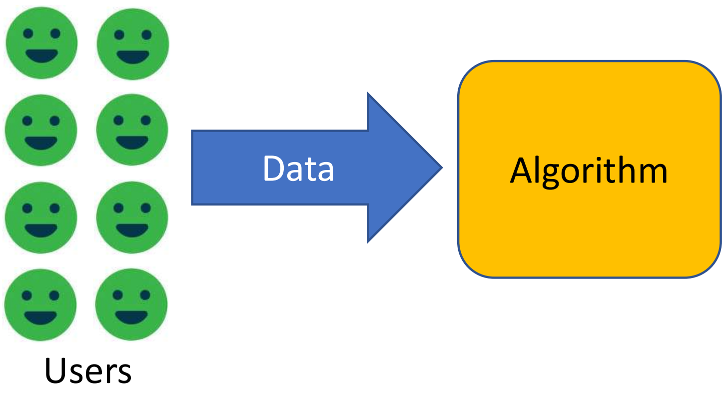

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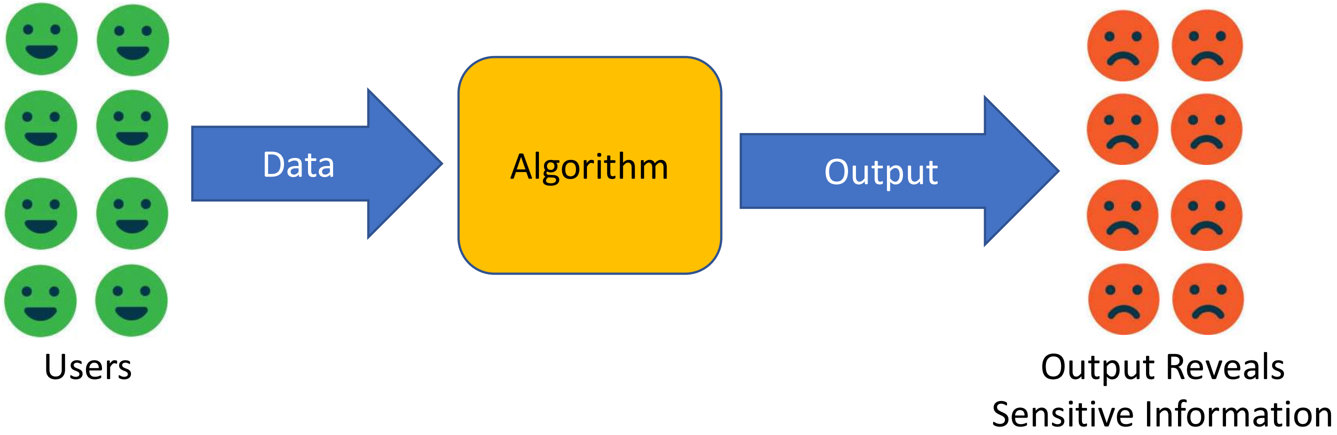

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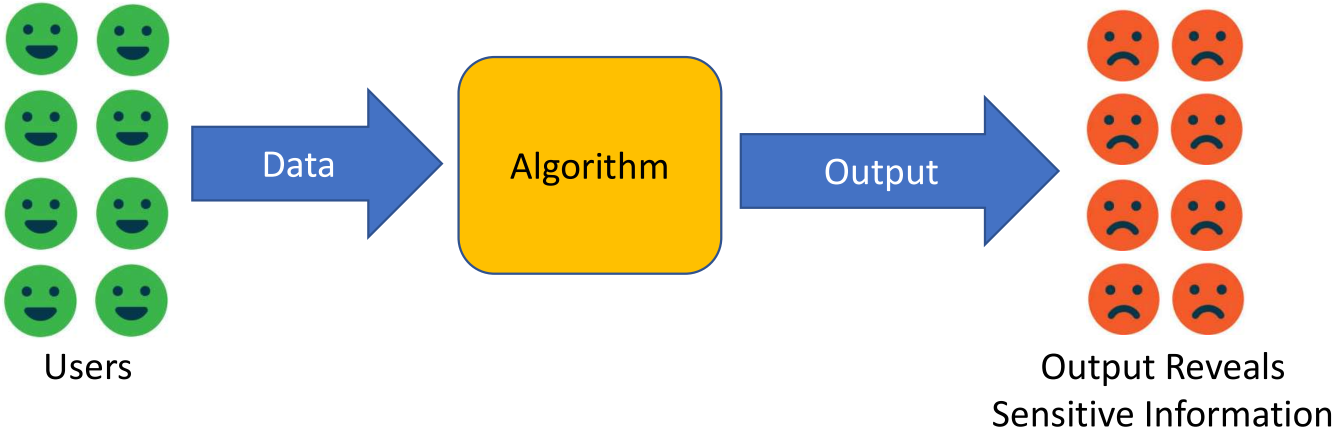
- Can we learn properties of data without revealing sensitive info?



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- This means that conditioned on seeing any output, we won't know if any individual data point X_i was in fact some other data point X_i'.

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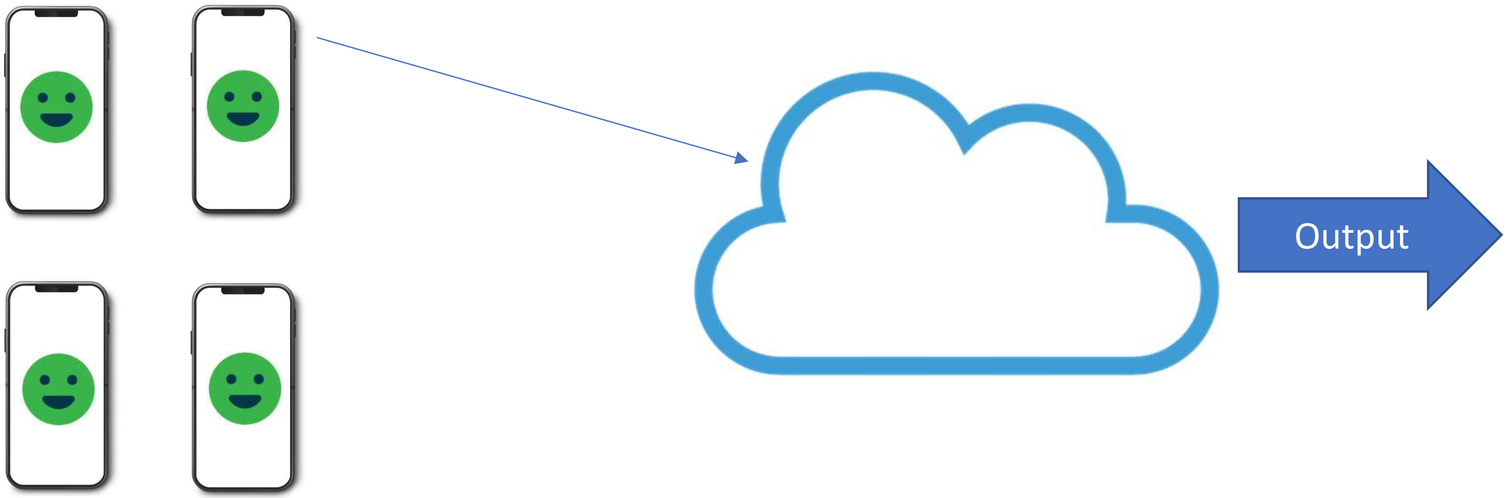

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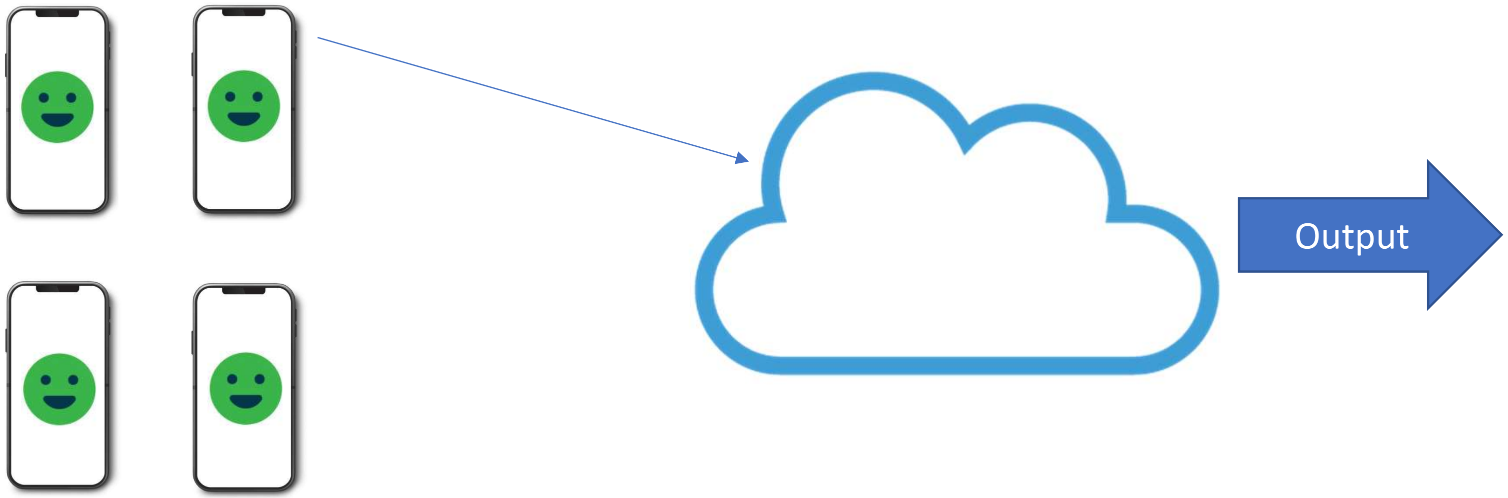

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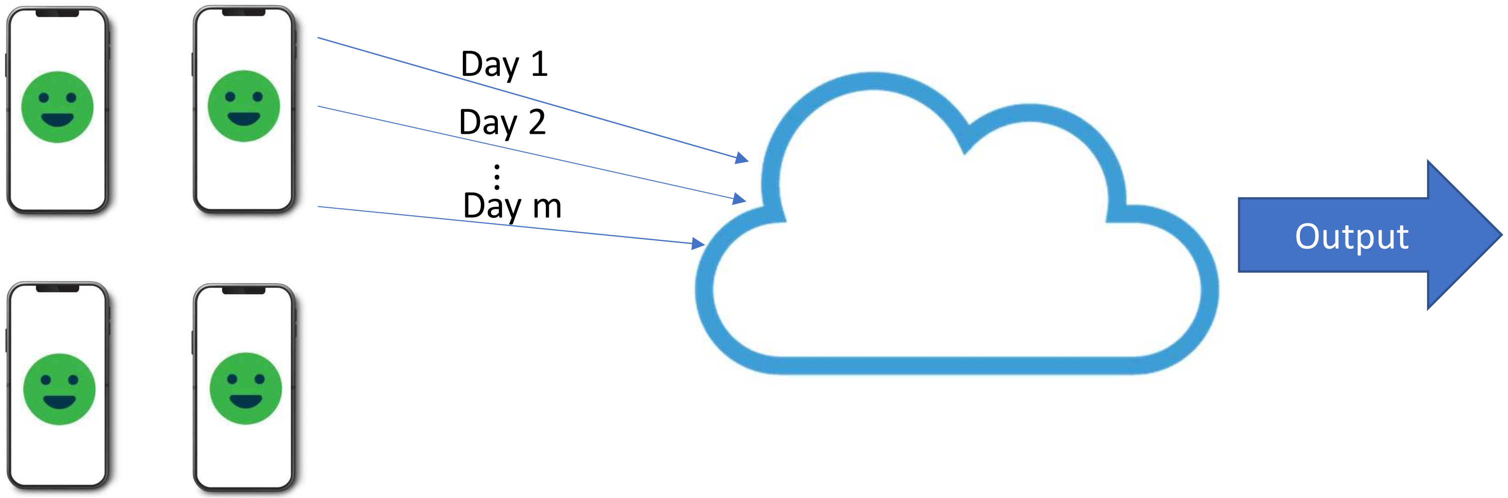

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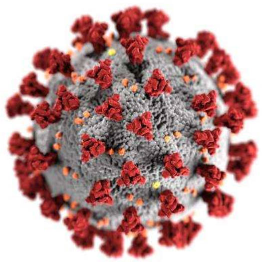

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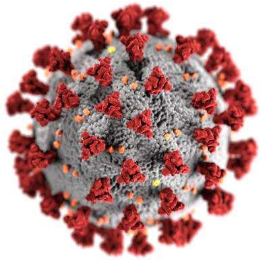

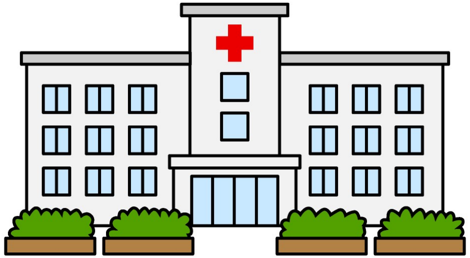

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- In our setting, we suppose each of n users outputs m i.i.d. samples from $\mathcal{D}$. Goal is to estimate μ with user-level DP.
- **Our result:** a private and low-error algorithm for mean estimation even with very few users (though each user may have many samples).

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- **Theorem:** Let $\mathcal{D}$ be a distribution over $\mathbb{R}^d$, concentrated in a ball of radius r (around an unknown location) and mean μ. Given $n = O\left(\frac{1}{\varepsilon} \cdot \log \frac{1}{\delta}\right)$ users and m samples per user, there is an (ε, δ)-user level DP algorithm that, if each sample were i.i.d. from $\mathcal{D}$, estimates μ up to error $r\sqrt{d/m}$.

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- Our algorithm runs in almost linear time in n and d. Our algorithm also works in the robust setting if even 49% of all users have all their samples corrupted (but the rest of the users have all samples intact).
- Algorithm can be applied to various learning problems (learning discrete distributions, stochastic convex optimization, etc.).

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- Answers a conjecture of Amin et al. (ICML 2019) asking about this user-sample tradeoff.
- Also improves over previous work of Liu et al. (NeurIPS 2020) and Levy et al. (NeurIPS 2021) which required $n \gg \sqrt{d \log \frac{1}{\delta}} / \varepsilon$.

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- Given $n = O\left(\frac{1}{\varepsilon} \cdot \log \frac{1}{\delta}\right)$ points $X_1, \dots, X_n$ in unknown ball of radius 1, approximate the ball up to error $\sqrt{d}$.

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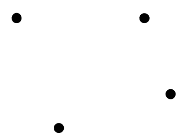

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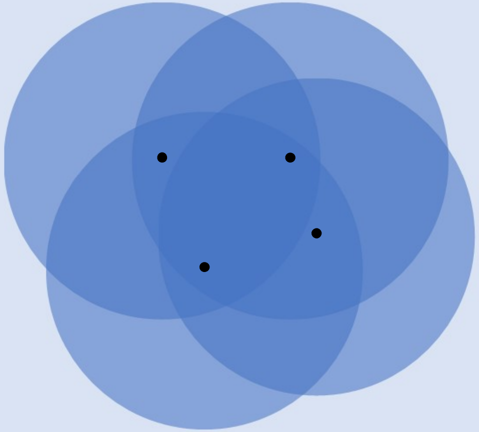

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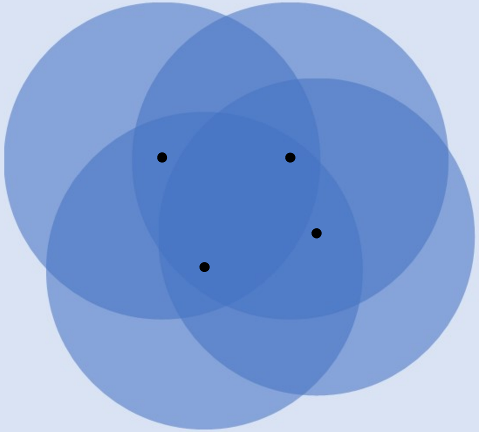

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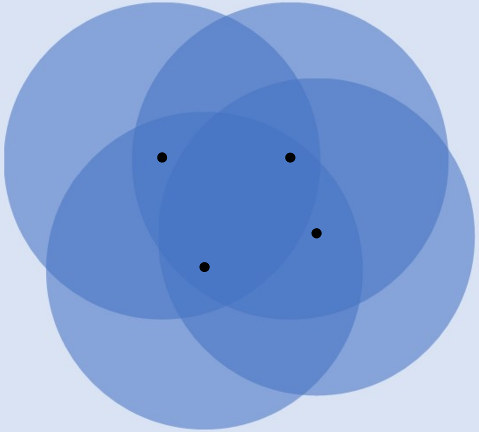

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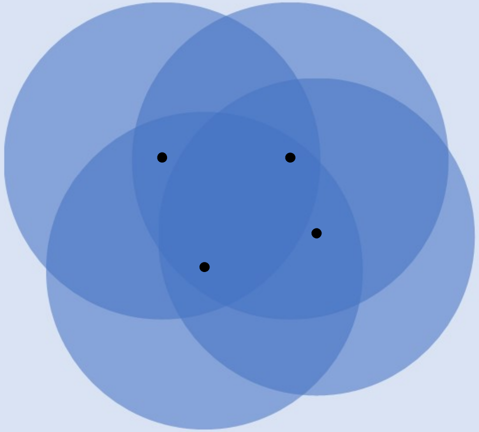

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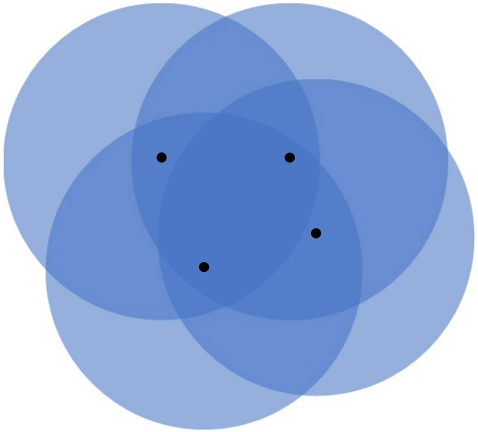

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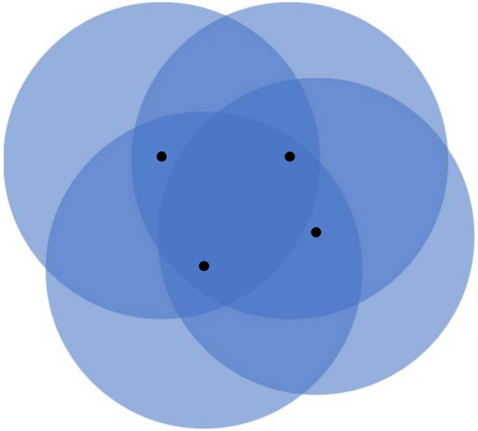

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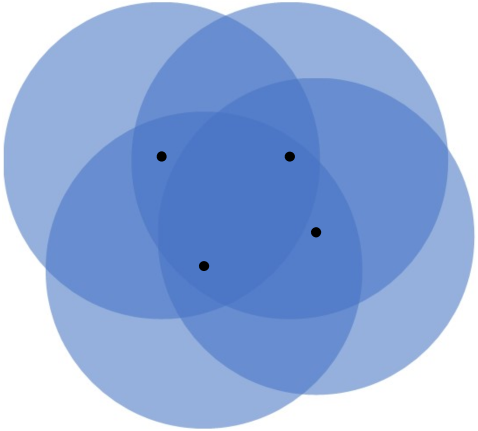

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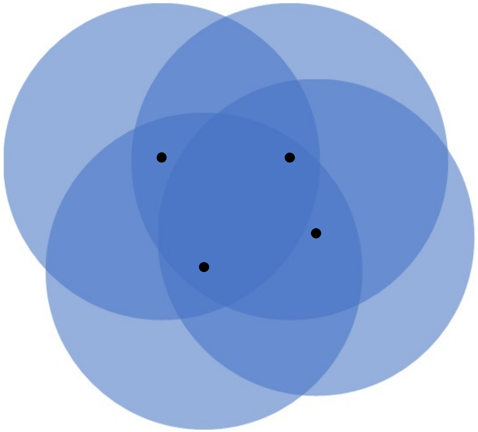

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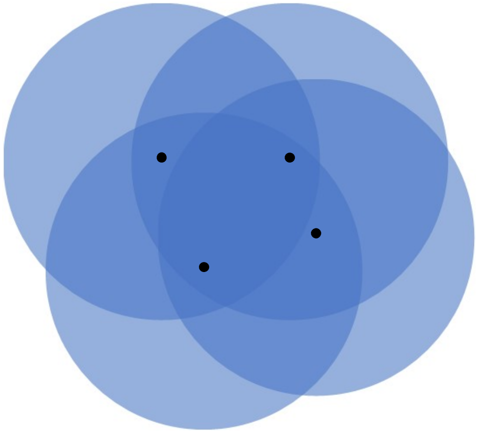

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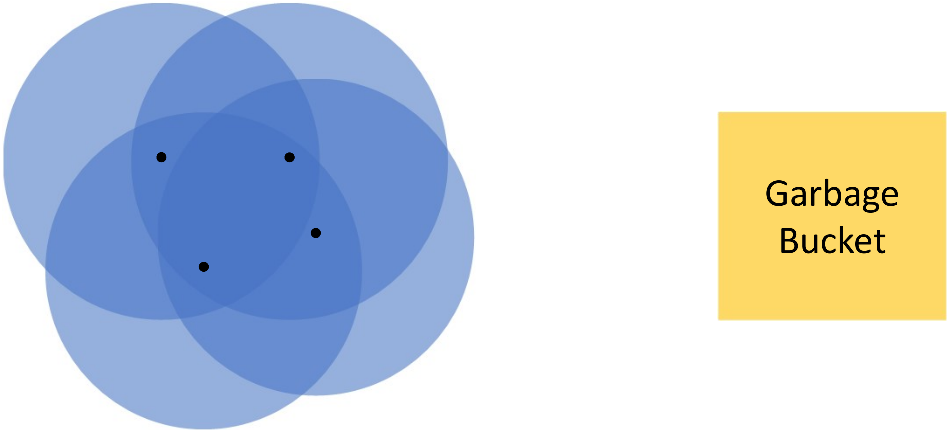

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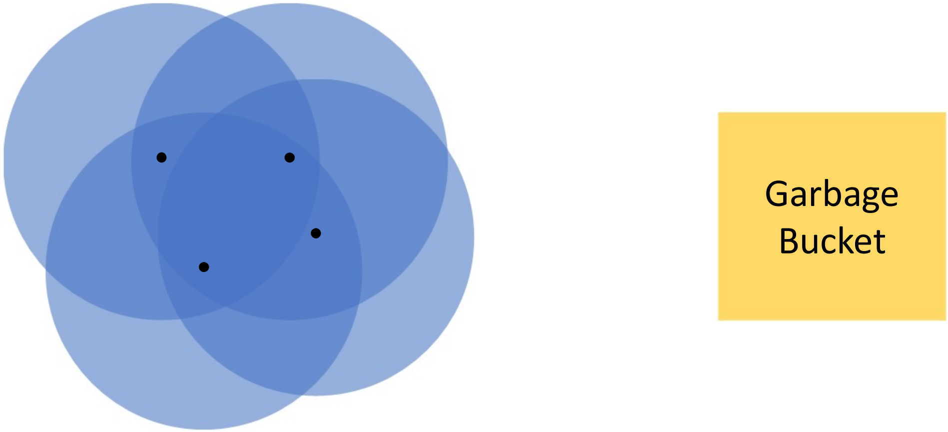

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- Need $$\underbrace{\frac{V}{\delta}}_{\text{volume of garbage bucket}} \ll \underbrace{e^{-\sqrt{\log n}} \cdot V}_{\text{volume of intersection}} \cdot \underbrace{e^{\varepsilon \cdot n}}_{\text{density}}.$$
- Holds if $n \geq O(\varepsilon^{-1} \log \delta^{-1})$.

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- If we reject, we keep trying until we accept a point p.

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- **Fix (attempt 1):** stop rejection sampling after some N rounds. Unfortunately, this method may no longer be private.
- **Fix:** stop after $Expo(N)$ rounds, for $N \approx e^{\sqrt{\log n}} = n^{o(1)}$. Allows for algorithm to be both fast and maintains privacy!

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- Our algorithm improves over previous methods, which required the number of users to depend at least on $\sqrt{d}$.
- Algorithm based on modifying exponential mechanism with garbage bucket, and rejection sampling techniques.

Thanks for attending!

Questions?