Towards Noise-adaptive, Problem-adaptive (Accelerated) Stochastic Gradient Descent

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Problem

Unconstrained minimization: finite-sum objective.

$$\min_{w \in \mathbb{R}^d} f(w) := \frac{1}{n} \sum_{i=1}^{n} f_i(w)$$

where n is the number of training examples.

- Smoothness and convexity: Each f_i is convex, differentiable and L_i-smooth, implying that f is L-smooth where $L := \max_i L_i$.
- Strong convexity: f is μ strongly-convex.

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- The two regimes require a different step-size choice (constant vs decreasing) and the convergence rate is not adaptive to the noise (σ^2) in the stochastic gradients.
- Require noise-adaptivity – one step-size sequence that can achieve the optimal rate in both the deterministic and stochastic settings without knowledge of σ^2.

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- smooth functions satisfying the PL condition using SGD with an exponentially decreasing sequence of step-sizes [Li et al., 2020]. Noise adaptive but requires knowledge of L.

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1. Can we design SGD step-sizes that are simultaneously (i) problem-adaptive and (ii) noise-adaptive – achieve the $\tilde{O}\left(\exp(-T/\kappa) + \frac{\sigma^2}{T}\right)$ rate without knowledge of L, μ or σ^2?
2. Can we obtain the accelerated $\tilde{O}\left(\exp(-T/\sqrt{\kappa}) + \frac{\sigma^2}{T}\right)$ rate?

Outline

- Problem 1: SGD with exponential step-sizes
  - Known smoothness
  - Online estimation of unknown smoothness
  - Offline estimation of unknown smoothness
- Problem 2: Accelerated SGD with exponential step-sizes
  - Known smoothness & strong-convexity
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- Experimental evaluation
- Conclusions and Future Work

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SGD with exponentially decreasing step-sizes

$$w_{k+1} = w_k - \underbrace{\gamma_k \alpha_k}_{:=\eta_k} \nabla f_{ik}(w_k) \tag{SGD}$$

where γ_k is the problem-dependent scaling term that captures the smoothness and α_k that controls the decay of the step-size.

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Exponentially decreasing step-sizes [Li et al., 2020]: $\alpha := \left[\frac{\beta}{T}\right]^{1/T} \leq 1$ for $\beta \geq 1$ and $\alpha_k := \alpha^k$.

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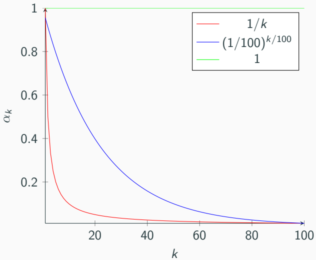
Lie between the constant and $1/k$ decreasing step-sizes, implying that for $k \in [T]$, $\alpha_k \in \left[\frac{1}{k}, 1\right]$.



Warm-up – known smoothness

- Assumption on the noise: $\sigma^2 := \mathbb{E}_i[f_i(w^*) - f_i^*]$.

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SGD with known smoothness

Assuming (i) convexity and L-smoothness of each f_i, (ii) μ strong-convexity of f, SGD with $\gamma_k = \frac{1}{L}$, $\alpha_k = \left(\frac{\beta}{T}\right)^{k/T}$ converges as,

$$\mathbb{E}\left\|w_{T+1} - w^*\right\|^2 \leq \left\|w_1 - w^*\right\|^2 c_2 \exp\left(-\frac{T}{\kappa}\frac{\alpha}{\ln(T/\beta)}\right) + \frac{8\sigma^2 c_2 \kappa}{\mu e^2}\frac{(\ln(T/\beta))^2}{\alpha^2 T},$$

where $\kappa = \frac{L}{\mu}$ and $c_2 = \exp\left(\frac{1}{\kappa} \cdot \frac{2\beta}{\ln(T/\beta)}\right)$.

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- Result can be concluded from Li et al. [2020], but we do not require the growth condition and use a different proof technique that helps handle unknown smoothness later.

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$$f_{ik}(w_k - \gamma_k \nabla f_{ik}(w_k)) \leq f_{ik}(w_k) - c\gamma_k \left\|\nabla f_{ik}(w_k)\right\|^2 .$$

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- When $\sigma \neq 0$, this method converges to a neighbourhood that depends on $\gamma_{\max}\sigma^2$.

Convergence of SGD with SLS

SGD with SLS – Upper Bound

Under the same assumptions, SGD with $\alpha_k = \left(\frac{\beta}{T}\right)^{k/T}$, γ_k as the largest step-size that satisfies $\gamma_k \leq \gamma_{\max}$ and the SLS condition with $c = 1/2$ converges as,

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$$+ \frac{2\sigma^2 c_1 \kappa' \ln(T/\beta)\,\left(\gamma_{\max} - \min\left\{\gamma_{\max}, \frac{1}{L}\right\}\right)}{e\alpha},$$

where $\kappa' := \max\left\{\frac{L}{\mu}, \frac{1}{\mu\gamma_{\max}}\right\}$, $c_1 = \exp\left(\frac{1}{\kappa'} \cdot \frac{2\beta}{\ln(T/\beta)}\right)$.

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- $O\left(\exp(-T/\kappa) + \sigma^2/T\right)$ convergence to a neighbourhood determined by σ^2 and initial estimation error $\left(\gamma_{\max} - \min\left\{\gamma_{\max}, \frac{1}{L}\right\}\right)$.

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When using T iterations of SGD to minimize the sum $f(w) = \frac{f_1(w)+f_2(w)}{2}$ of two one-dimensional quadratics, $f_1(w) = \frac{1}{2}(w-1)^2$ and $f_2(w) = \frac{1}{2}\left(2w + {}^1\!/_2\right)^2$, setting the step-size using SLS with $\gamma_{\max} \geq 1$ and $c \geq {}^1\!/_2$, any convergent sequence of α_k results in convergence to a neighbourhood of the solution. Specifically, if $w_1 > 0$, then,

$$\mathbb{E}(w_T - w^*) \geq \min\left(w_1, \frac{3}{8}\right).$$

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What if estimate the smoothness offline – without any correlation between γ_k and i_k?

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where $c_2 = \exp\left(\frac{1}{\kappa} \frac{2\beta}{\ln(T/\beta)}\right)$, $[x]_+ = \max\{x, 0\}$, $G = \max_{j \in [k_0]}\{f(w_j) - f^*\}$.

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- Ensures convergence to the minimizer, but the rate is slowed down proportional to ν.
- For polynomial α_k sequences, Moulines and Bach [2011] show an $\exp(\nu)$ dependence on the rate $\implies$ exponential step-sizes are more robust towards misspecification.

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$$w_{k+1} - w^* = (w_1 - w^*) \prod_{i=1}^{k} (1 - \nu\alpha_i).$$

After $k' := \frac{T}{\ln(T/\beta)} \ln\left(\frac{\nu}{3}\right)$ iterations, we have that

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Accelerated SGD with exponentially decreasing step-sizes

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Assumption on the noise: $\mathbb{E}_i \left\|\nabla f_i(w)\right\|^2 \leq \rho \left\|\nabla f(w)\right\|^2 + \sigma^2$

$$\begin{aligned} y_k &= w_k + b_k \left(w_k - w_{k-1}\right), \\ w_{k+1} &= y_k - \gamma_k \alpha_k \nabla f_{ik}(y_k). \end{aligned} \tag{ASGD}$$

where $\gamma_k = \frac{1}{\rho L}$, $\alpha_k = \left(\frac{\beta}{T}\right)^{k/T}$, $r_k = \sqrt{\frac{\mu}{\rho L}} \left(\frac{\beta}{T}\right)^{k/2T}$ and $b_k = \frac{(1-r_{k-1})\, r_{k-1}\, \alpha}{r_k + r_{k-1}^2\, \alpha}$.

Accelerated SGD with exponentially decreasing step-sizes

Assumption on the noise: $\mathbb{E}_i \left\|\nabla f_i(w)\right\|^2 \leq \rho \left\|\nabla f(w)\right\|^2 + \sigma^2$

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Equivalent to Nesterov acceleration if we use a deterministic gradient $\nabla f(y_k)$ and $\gamma_k = \gamma = \frac{1}{L}$ and $\alpha_k = 1$ for all k.

Convergence of Accelerated SGD – Known smoothness & strong-convexity

Convergence of ASGD

Under the same assumptions as before and (iii) the growth condition on the stochastic gradients, ASGD with $w_1 = y_1$, $\gamma_k = \frac{1}{\rho L}$, $\alpha_k = \left(\frac{\beta}{T}\right)^{k/T}$, $r_k = \sqrt{\frac{\mu}{\rho L}}\left(\frac{\beta}{T}\right)^{k/2T}$ and $b_k = \frac{(1-r_{k-1})\, r_{k-1}\, \alpha}{r_k + r_{k-1}^2\, \alpha}$ converges as,

$$\mathbb{E}[f(w_{T+1}) - f^*] \leq 2c_3 \exp\left(-\frac{T}{\sqrt{\kappa\rho}} \frac{\alpha}{\ln(T/\beta)}\right) \mathbb{E}[f(w_1) - f^*] + \frac{2\sigma^2 c_3}{\rho\mu e^2} \frac{(\ln(T/\beta))^2}{\alpha^2 T},$$

where $c_3 = \exp\left(\frac{2\beta}{\sqrt{\rho\kappa}\ln(T/\beta)}\right)$.

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- In the deterministic setting, $\rho = 1$ and $\sigma = 0$, and ASGD is near-optimal.

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- Aybat et al. [2019] use a more complicated algorithm and prove this rate when $T \geq 2\sqrt{\kappa}$.

ASGD with offline estimation of the smoothness & strong-convexity

- Assume $\gamma_k = \gamma = \frac{1}{\rho \tilde{L}} = \frac{\nu_L}{\rho L}$ and $\tilde{\mu} = \nu_\mu \mu$ where $\nu_\mu \leq 1$.

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Convergence of ASGD

Under the same assumptions and $\nu = \nu_L \nu_\mu \leq \rho\kappa$, ASGD with $w_1 = y_1$, $\gamma_k = \frac{1}{\rho \tilde{L}} = \frac{\nu_L}{\rho L}$, $\alpha_k = \left(\frac{\beta}{T}\right)^{k/T}$, $\tilde{\mu} = \nu_\mu \mu \leq \mu$, $r_k = \sqrt{\frac{\nu}{\rho\kappa}} \left(\frac{\beta}{T}\right)^{k/2T}$ and $b_k = \frac{(1-r_{k-1})\, r_{k-1}\, \alpha}{r_k + r_{k-1}^2\, \alpha}$ converges as,

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$$+ \frac{2c_3 (\ln(T/\beta))^2}{e^2 \alpha^2 \mu T} \left[\frac{\sigma^2}{\rho} + G^2 \min\{\frac{k_0}{T}, 1\}\right] \max\{\frac{\nu_L}{\nu_\mu}, \nu_L^2\},$$

where $c_3 = \exp\left(\frac{1}{\sqrt{\rho\kappa}} \frac{2\beta}{\ln(T/\beta)}\right)$, $k_0 := \lfloor T \frac{[\ln(\nu_L)]_+}{\ln(T/\beta)} \rfloor$, $G = \max_{j\in[k_0]} \|\nabla f(y_j)\|$.

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- Implies an $\tilde{O}\left(\exp\left(\frac{-T\sqrt{\min\{\nu,1\}}}{\sqrt{\kappa\rho}}\right) + \left[\frac{\sigma^2 + G^2 [\ln(\nu_L)]_+}{T}\right] \max\{\frac{\nu_L}{\nu_\mu}, \nu_L^2\}\right)$ rate.

Outline

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  - Offline estimation of unknown smoothness
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Experimental evaluation

- **Conservative decorrelated SLS**: Line-search starting from γ_{k-1} (with $\gamma_0 = \gamma_{\max}$) for a random or previously sampled function (j_k), find the largest step-size γ_k that satisfies

$$f_{j_k}(w_k - \gamma_k \nabla f_{j_k}(w_k)) \leq f_{j_k}(w_k) - c\gamma_k \left\| \nabla f_{j_k}(w_k) \right\|^2,$$

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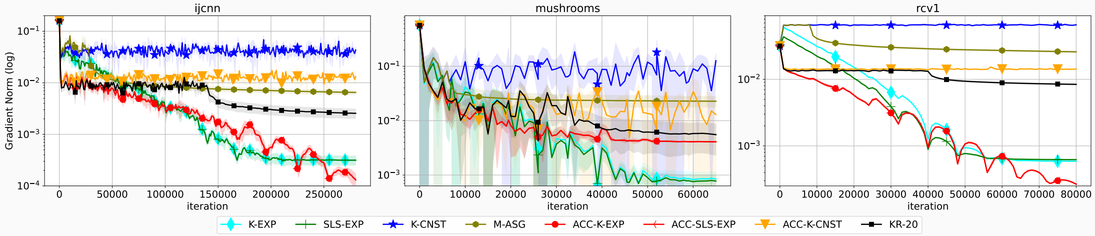

Figure 1: Regularized logistic regression

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Other results and Future work

- Results for strongly star-convex functions [Hinder et al., 2020].
- Effect of batch-size for all results.
- Result showing that no polynomial step-size can achieve the desired noise-adaptive rate.
- Exponential step-sizes do not seem to be noise-adaptive for convex functions (without strong-convexity) [Upper-bound]. Results showing that it is unlikely any exponential/polynomial step-size will be noise-adaptive in this case.

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- Algorithm without any price of misestimation.
- Step-size schemes that are noise-adaptive for convex functions.

Questions?

Paper: https://arxiv.org/abs/2110.11442
Code: https://github.com/R3za/expsls
Contact: vaswani.sharan@gmail.com

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