Fast Projection Onto Convex Smooth Constraints

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Euclidean Projection Problem:

$$\min_{x \in \mathbb{R}^n} \quad \|x_0 - x\|^2$$
$$\textbf{s.t.} \quad x \in \mathcal{K}$$

$$\mathcal{K} := \{x \in \mathbb{R}^n : h_i(x) \leq 0, \forall i = 1,\ldots,m\}$$

$h_i : \mathbb{R}^n \to \mathbb{R}$ are smooth and convex

Set	Approach	Computational complexity	Required oracle
$\mathcal{K}$ has a well defined self-concordant barrier	Interior point method (primal-dual IPM)	$O\left(n^3 \log \frac{n}{\varepsilon}\right)$	Second-order (Hessian)
$\mathcal{K} = \cap_{i=1}^m \mathcal{K}_i$	Alternating Directions Method of Multipliers (consensus ADMM)	$O\left(\frac{1}{\varepsilon}\right)$	Projection oracles for $\mathcal{K}_i$ separately

Can we proceed faster in the high dimensional case? $O\left(nm^{2.5}\log^2(m/\varepsilon) + m^{3.5}\log(m/\varepsilon)\right)$

Warm-up:
Projection onto a single smooth constraint

Projection problem:

$$\begin{array}{ll} \min & \|x_0 - x\|^2 \\ \textbf{s.t.} & h(x) \leq 0 \end{array}$$

The Lagrangian:

$$L(x, \lambda) = \|x_0 - x\|^2 + \lambda h(x)$$

Strong duality:

$$\min_{x \in \mathbb{R}^d} \max_{\lambda \geq 0} L(x, \lambda) = \max_{\lambda \geq 0} \min_{x \in \mathbb{R}^d} L(x, \lambda)$$

Warm-up:
Projection onto a single smooth constraint

Dual problem: $\max_{\lambda \geq 0} d(\lambda)$

Dual function: $d(\lambda) = \min_{x \in \mathbb{R}^d} \|x_0 - x\|^2 + \lambda h(x)$

Smooth & 2-strongly-convex

Due to strong convexity, x^*_λ is unique

$$d(\lambda) = \|x_0 - x^*_\lambda\|^2 + \lambda h(x^*_\lambda)$$

Gradient of the dual: $\nabla d(\lambda) = h(x^*_\lambda)$

$\max_{\lambda \geq 0} d(\lambda)$ ← solve using bisection ← $\nabla d(\lambda) = h(x^*_\lambda)$ ← Find x^*_λ using Accelerated Gradient Descent (AGD) Nesterov (1998)

smooth, concave on λ 1-dimensional

$O\left(\log \frac{1}{\varepsilon}\right)$

$O\left(n \log \frac{1}{\tilde{\varepsilon}}\right)$

Projection onto m smooth constraints

Projection problem:

$$\min \|x_0 - x\|^2$$
$$\textbf{s.t. } \mathbf{h}(x) \leq 0$$
$$\mathbf{h}(x)_i = h_i(x), i = 1,\ldots,m$$

Dual problem:

$$\max_{\lambda \in \mathbb{R}_+^m} d(\lambda)$$

$$d(\lambda) = \min_{x \in \mathbb{R}^d} \|x_0 - x\|^2 + \lambda^T \mathbf{h}(x)$$

Gradient of the dual objective:

$$\nabla d(\lambda) = \mathbf{h}(x_\lambda^*)$$

Fast Projection Method

$$\max_{\lambda \in \mathbb{R}_+^m} d(\lambda)$$

m-dimensional concave problem

← solve using the cutting plane scheme ←

$O\left(m^{3.5} \log \frac{1}{\varepsilon}\right)$

$$\nabla d(\lambda) = \mathbf{h}(x_\lambda^*)$$

← Find x_λ^* using AGD

Nesterov (1998)

$O\left(n \log \frac{1}{\tilde{\varepsilon}}\right)$

Welcome to reading our paper!