Besov Function Approximation and Binary Classification on Low-Dimensional Manifolds Using Convolutional Residual Networks

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Introduction

Deep neural networks (NN) have demonstrated impressive performance in

- Computer vision [Krizhevsky et al., 2012]
- Natural language processing [Graves et al., 2013; Young et al., 2018; Wu et al., 2016]
- Health care [Miotto et al., 2018; Jiang et al., 2017]
- Bioinformatics [Alipanahi et al., 2015; Zhou & Troyanskaya, 2015]

Existing theories for FNNs and CNNs

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- Feedforward neural networks (FNN) [Cybenko 1989; Hamers & Kohler 2006; Kohler & Mehnert 2011; Lu et al. 2017; Yarotsky 2017; Lee et al. 2017; Suzuki 2019]
- Convolutional neural networks (CNN) [Petersen & Voigtlaender, 2020; Zhou 2020a, 2020b, Oono & Suzuki 2019.]

Existing theories for FNNs and CNNs

Most of the existing work on FNNs and CNNs

- Are cursed by dimensionality:
  - To approximate a C^s function in $\mathbb{R}^D$ with accuracy ε, the network size is of $O(\varepsilon^{-D/s})$.
- Study Hölder or Sobolev functions.
  - [Suzuki 2019] studied Besov functions $B^{s+\alpha}_{p,q}$

$$W^{s+\alpha,\infty} = \mathcal{H}^{s,\alpha} \subseteq B^{s+\alpha}_{\infty,\infty} \subseteq B^{s+\alpha}_{p,q}$$

for any $0 < p, q \leq \infty, s \in \mathbb{N}$ and $\alpha \in (0, 1]$.

Existing theories for FNNs and CNNs

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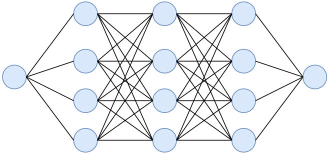
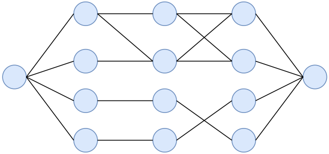
- Have a cardinality constraint



Without the cardinality constraint



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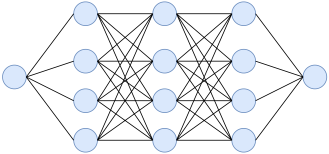

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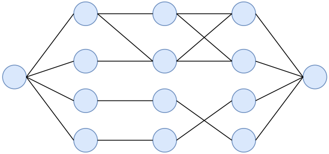
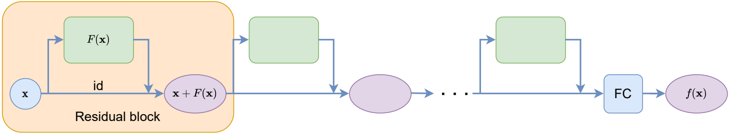


With a cardinality constraint

Training networks with the cardinality constraint needs substantial efforts [Han et al. 2015, 2016; Blalock et al. 2020].

ConvResNets

The convolutional residual network (ConvResNet) is a special CNN with skip-layer connections.



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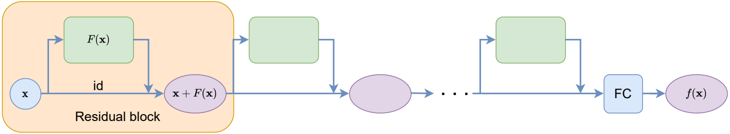


- Approximation theory of ConvResNets for Hölder functions is developed by [Oono & Suzuki 2019].

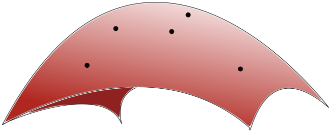
Properties:

- No cardinality constraints
- Cursed by dimensionality

Our work

Our work

- We assume the data or target functions are located on a d-dimensional manifold $\mathcal{M}$ embedded in $\mathbb{R}^D$ with $d < D$.



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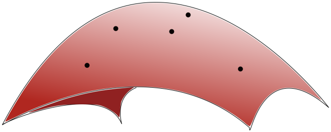

- We analyze the performance of ConvResNets on
  - Besov function approximation
  - Binary classification with the logistic loss

Besov function approximation

Theorem

Assume $0 < p, q \leq \infty$, $d/p + 1 \leq s < \infty$. *Given* $\varepsilon \in (0, 1)$ *and under mild assumptions, we construct a ConvResNet architecture. For any* $f^* \in B^s_{p,q}(\mathcal{M})$, *if the weight parameters of this ConvResNet are properly chosen, it gives rises to* $\bar{f}$ *satisfying*

$$\|\bar{f} - f^*\|_{L^\infty} \leq \varepsilon.$$

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Remarks:

- There is no cardinality constraint.
- The network size is of $O(\varepsilon^{-d/s})$, and only weakly depends on D.

Binary classification

Problem settings:

- We are given a set of data $\{\mathbf{x}_i, y_i\}_{i=1}^n, \mathbf{x}_i \in \Omega$ in $\mathbb{R}^D$ and $y_i \in \{-1, 1\}$ follows the Bernoulli-type distribution

$$\mathbb{P}(y = 1|\mathbf{x}) = \eta(\mathbf{x}),\ \mathbb{P}(y = -1|\mathbf{x}) = 1 - \eta(\mathbf{x}).$$

- Learn a classifier using ConvResNets by minimizing the empirical logistic loss

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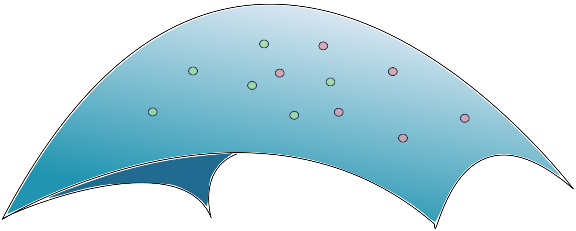
A low dimension manifold model for inputs:

- Assume $\{\mathbf{x}_i\}_{i=1}^n$ are located on a d-dimensional manifold $\mathcal{M}$ embedded in $\mathbb{R}^D$.



Binary classification

Theorem

Under the settings of the previous theorem, assume $\eta \in B^s_{p,q}(\mathcal{M})$. *Let* f^*_ϕ *be the minimizer of the population logistic risk. we construct a ConvResNet architecture with which minimizing the empirical logistic risk gives rise to* $\widehat{f}_{\phi,n}$ *with the following excess risk bound*

$$\mathbb{E}(\mathcal{E}_\phi(\widehat{f}_{\phi,n}, f^*_\phi)) \leq Cn^{-\frac{s}{2s+2(s\vee d)}} \log^4 n,$$

where $\mathcal{E}_\phi(\widehat{f}_{\phi,n}, f^*_\phi)$ *denotes the excess logistic risk of* $\widehat{f}_{\phi,n}$ *against* f^*_ϕ *and* C *is a constant independent of* n.

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Remarks:

- Our result gives a faster rate depending on d instead of D
- ConvResNets are adaptive to the intrinsic dimension of data sets

Thank You