Fast Sketching of Polynomial Kernels of Polynomial Degree

ICML 2021 speaker: Lichen Zhang (CMU)

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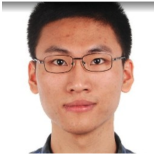

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- Additionally, we consider the high-dimensional, or over-parametrized setting, where $d \gg n$. This is of particular interest in NLP, biology and training over-parametrized neural networks.

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- Polynomial kernel of degree q is defined as $P_q(x, y) = \langle x, y \rangle^q$.

Intuition

- Instead of computing P_q directly, we compute $P_q = \Phi^\top \Phi$, where $\Phi \in \mathbb{R}^{d^q \times n}$ is defined using a *lifting function* $\phi : \mathbb{R}^d \to \mathbb{R}^{d^q}$ with $\phi(x)_{i_1, i_2, \ldots, i_q} = x^{\otimes q} = x_{i_1} x_{i_2} \ldots x_{i_q}$ for $i_1, i_2, \ldots, i_q \in \{1, 2, \ldots, d\}$.

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- $\Phi = X^{\otimes q} \in \mathbb{R}^{d^q \times n}$.
- Goal: approximate $X^{\otimes q}$ efficiently without explicitly forming it. Typically, it involves looking for a matrix $\Pi \in \mathbb{R}^{s \times d^q}$ with $s \ll d^q$ and forming $\Pi X^{\otimes q}$ very fast.

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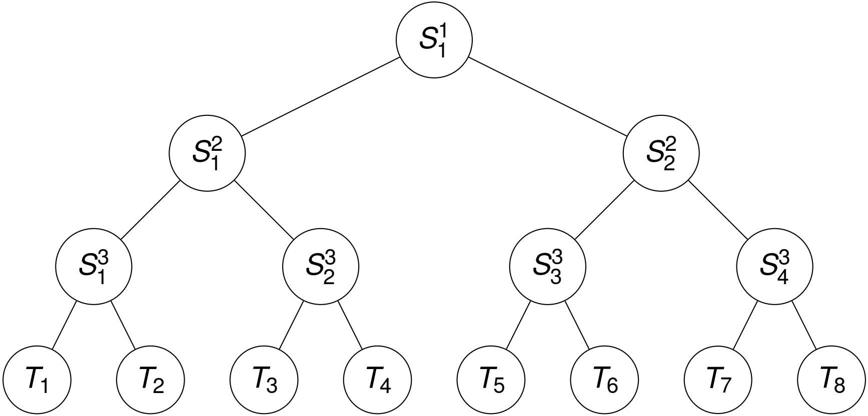
  - Adaptive Leverage score sampling (Woodruff and Zandieh, 20).
- State-of-the-art are oblivious sketching and adaptive leverage score sampling.

Oblivious sketching

Oblivious sketching: view the computation of $x^{\otimes q}$ as a binary tree with q nodes.



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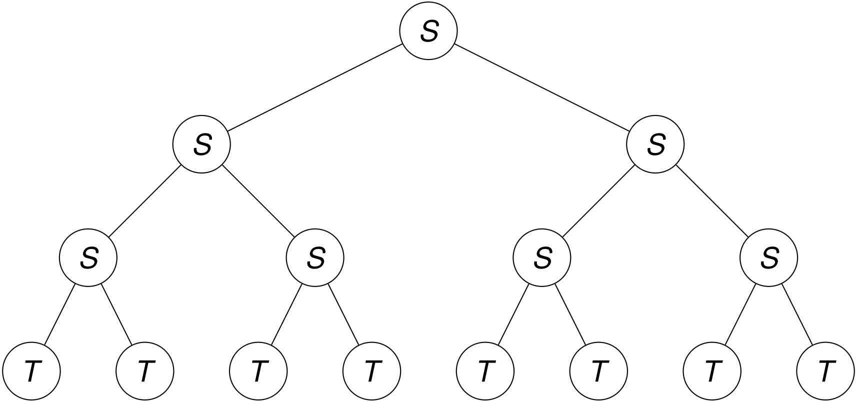
- Can we reduce the amount of randomness and improve the running time of oblivious sketching?

Our approach: constructing the tree with limited randomness

Instead of using q different random sketches, we just use *2* of them.



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- However, since we use only a constant amount of randomness, the guarantee of $\Pi X^{\otimes q}$ is weaker than oblivious sketching: we can only spectrally approximate $X^{\otimes q}$.
- Nevertheless, it can be combined with other sketching techniques to achieve the overall strong guarantee of approximate matrix product as well and in a faster way.

Applications

- Our polynomial kernel approximation algorithm can be applied to various settings, such as approximate a Gaussian kernel, our algorithm gives the fastest result in over-parametrized and unregularized setting.

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  - This leads to fast algorithm for approximating arc-cosine kernels and neural tangent kernels.

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- Solve kernel ridge regression by composing with another sketch with even smaller size.

Thanks for attending the talk!

- If you have any questions regarding our paper, feel free to contact the authors via email:

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