Commutative Lie Group VAE for Disentanglement Learning (Long Talk)

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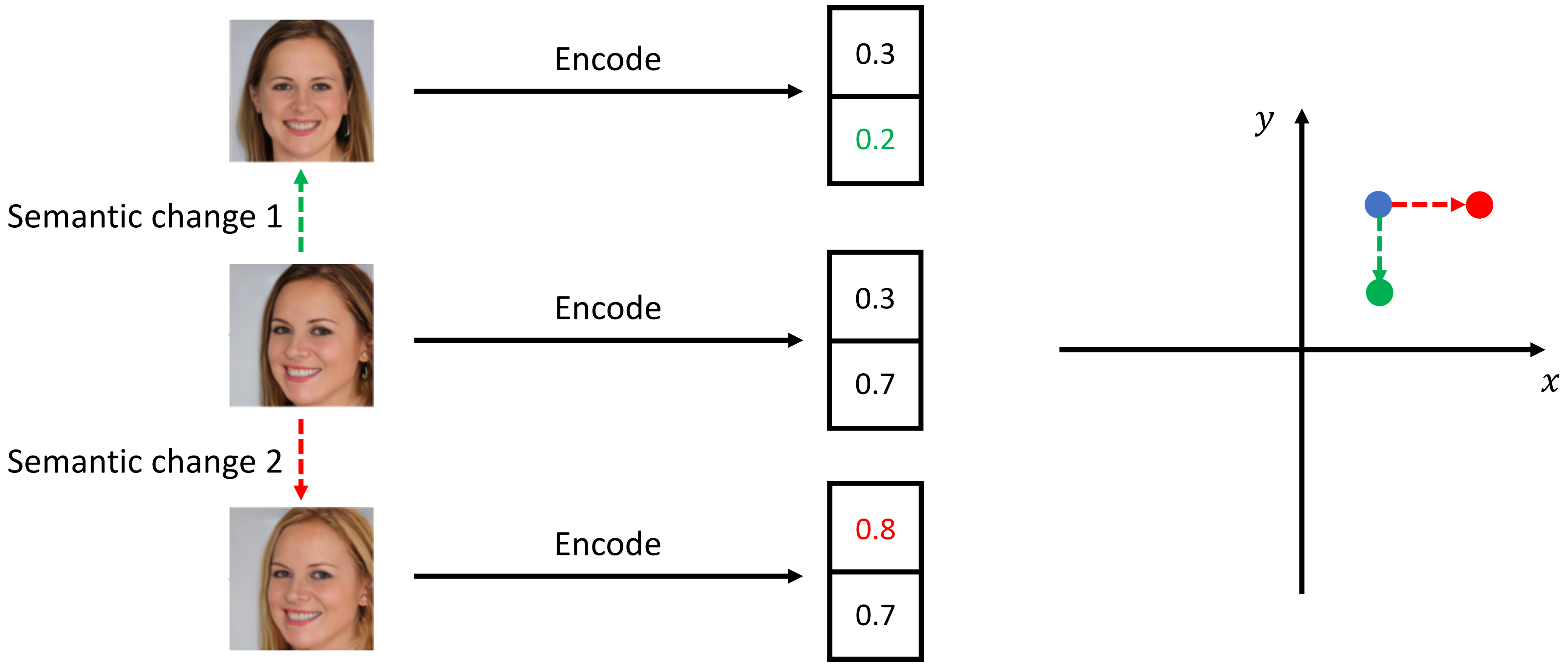
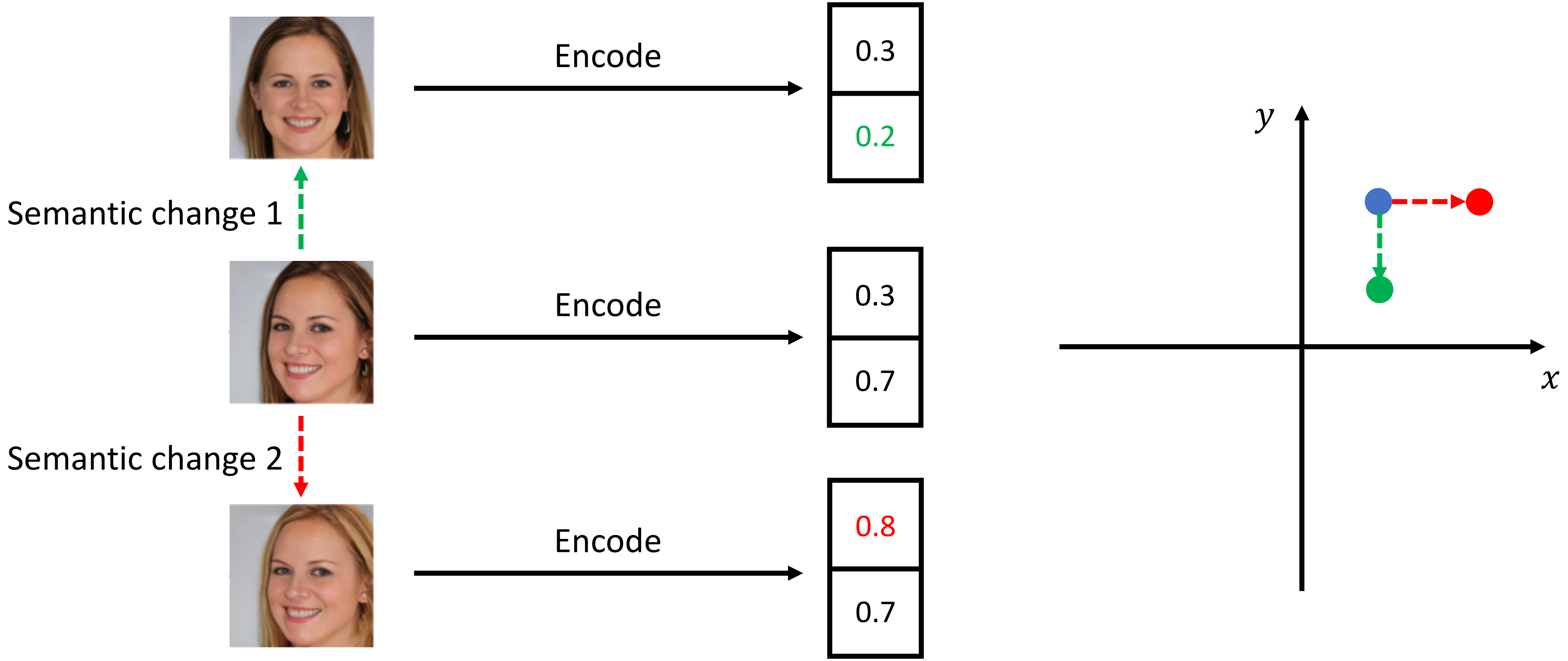
Rethinking Disentanglement Learning



Traditional disentangled representation.

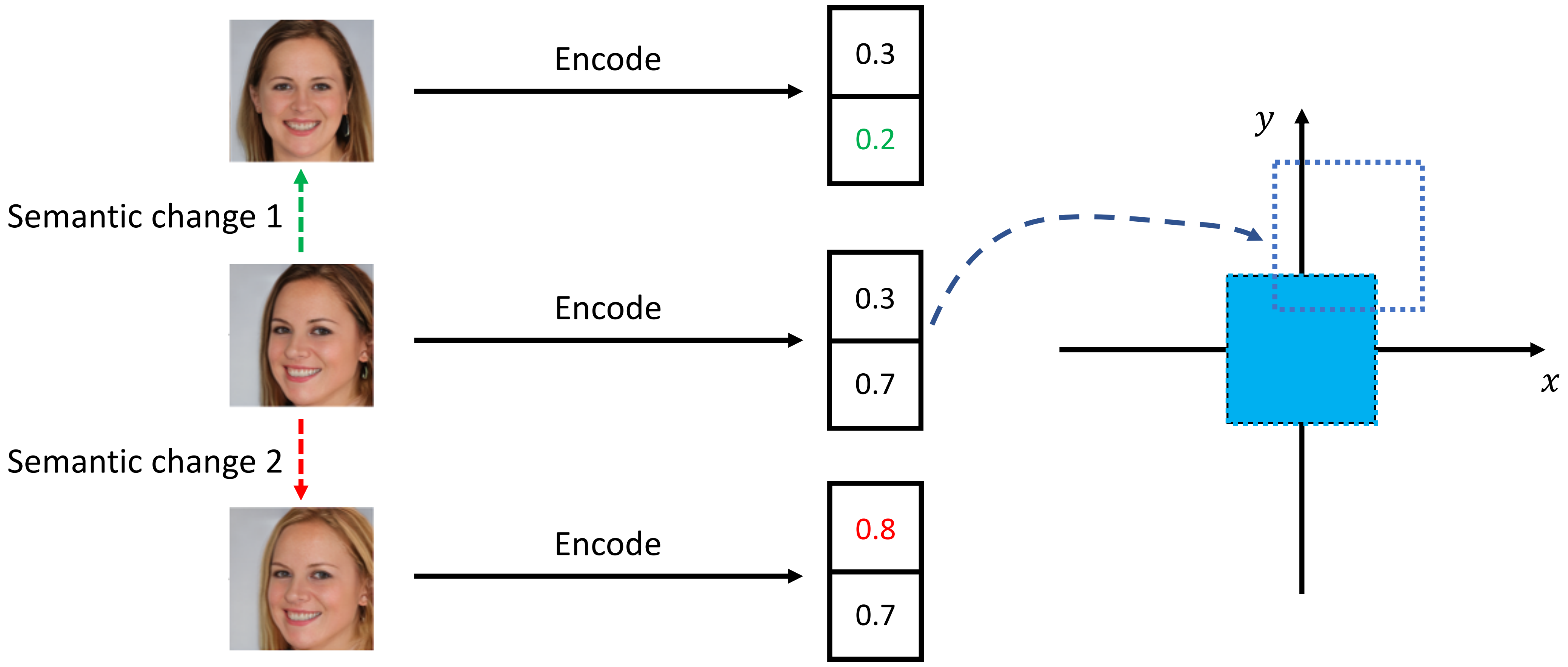
We view each embedded sample as a point.

Rethinking Disentanglement Learning



However, we can also view it as space translation.

Rethinking Disentanglement Learning



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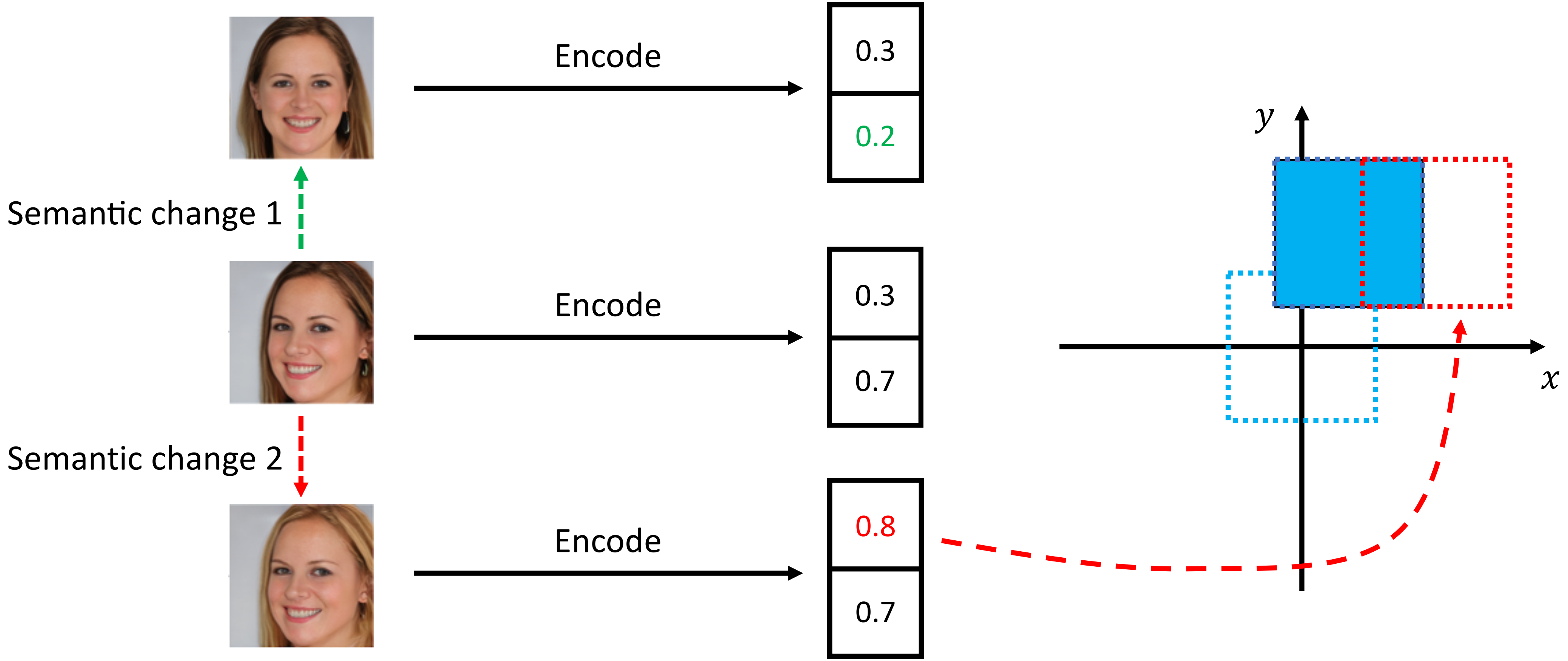
Rethinking Disentanglement Learning



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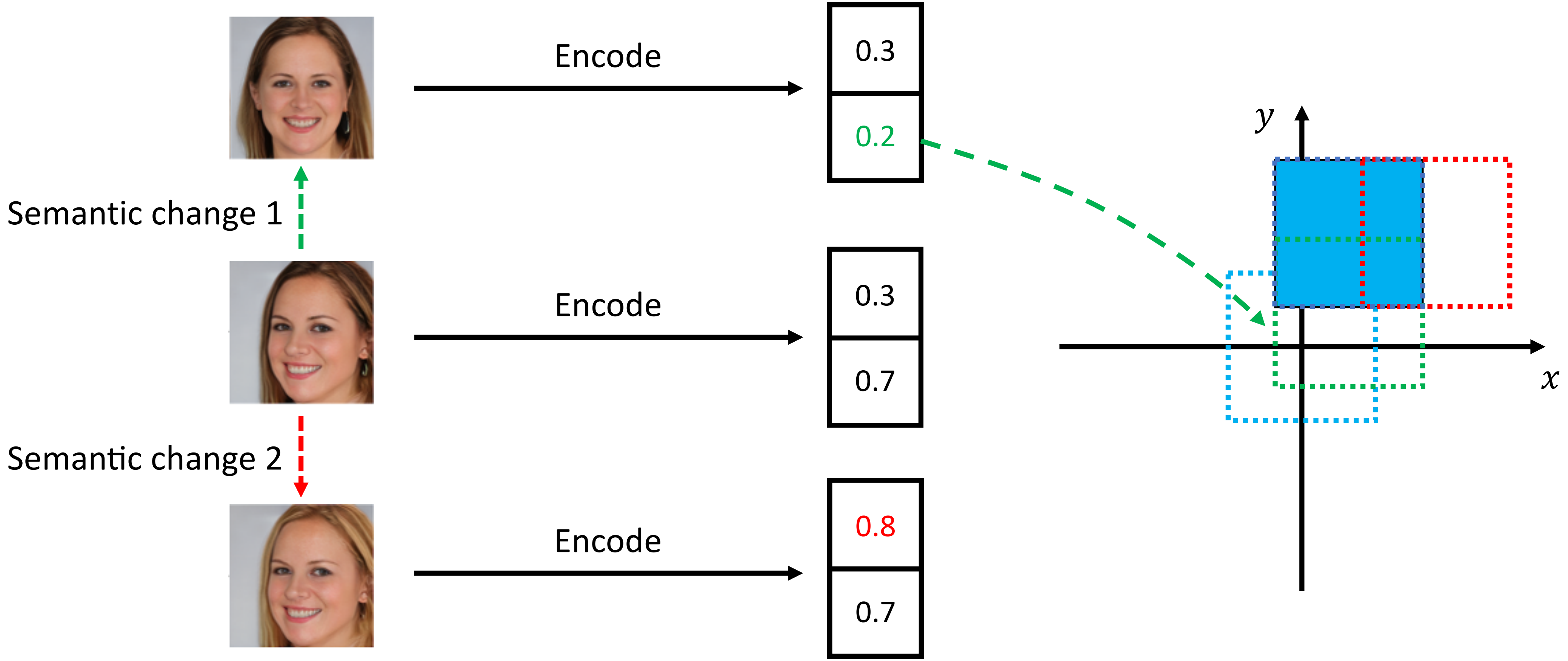
Rethinking Disentanglement Learning



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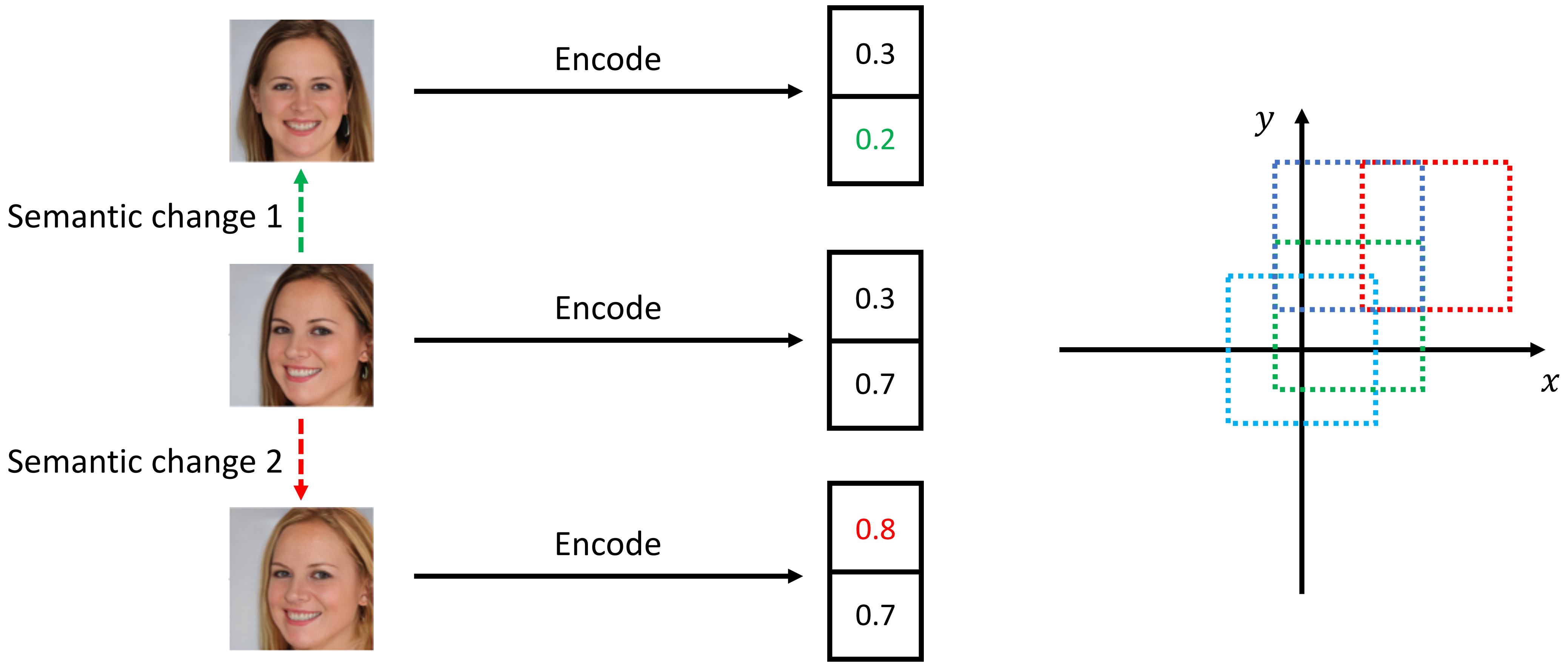
Rethinking Disentanglement Learning



However, we can also view it as space translation.

Image variation $\xleftrightarrow{\text{Equivariance}}$ Latent space translation

Rethinking Disentanglement Learning

Image variation $\xleftrightarrow{\text{Equivariance}}$ Latent space $\boxed{\text{translation}}$

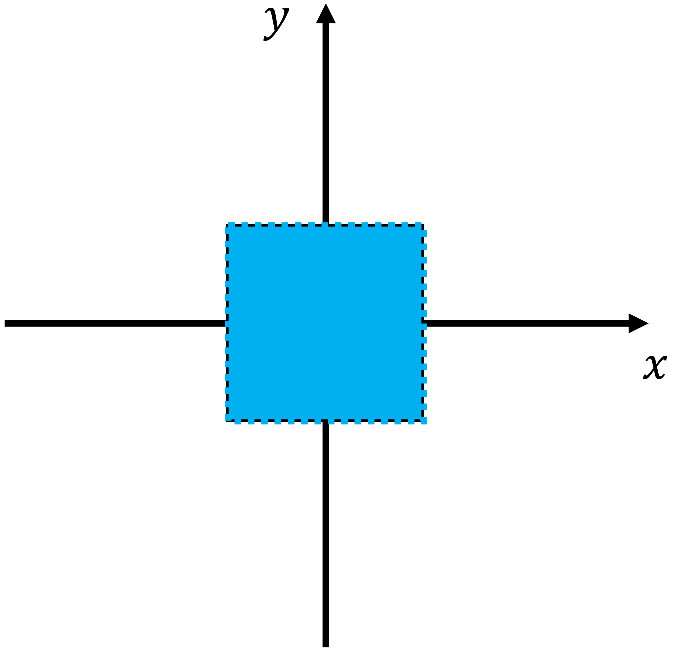
Why translation?

Other transformations are possible.

Rethinking Disentanglement Learning

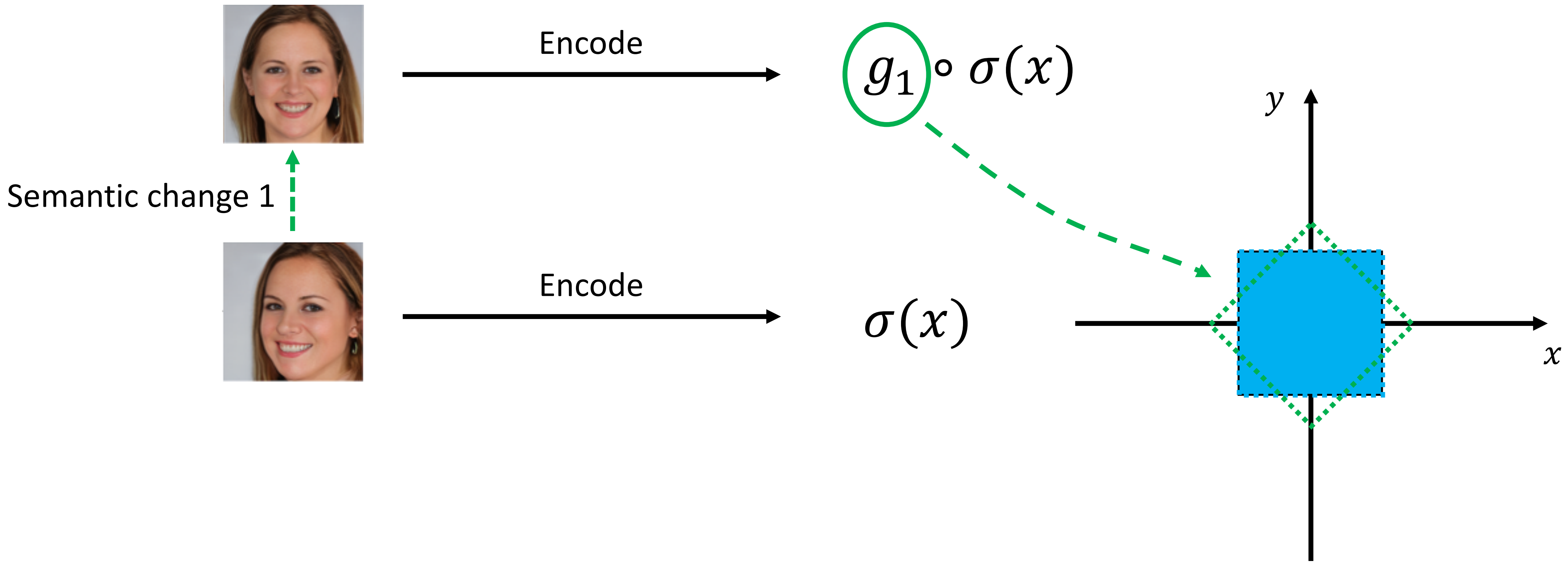
Encode

$\sigma(x)$



Suppose we encode an image as a state of the latent space.

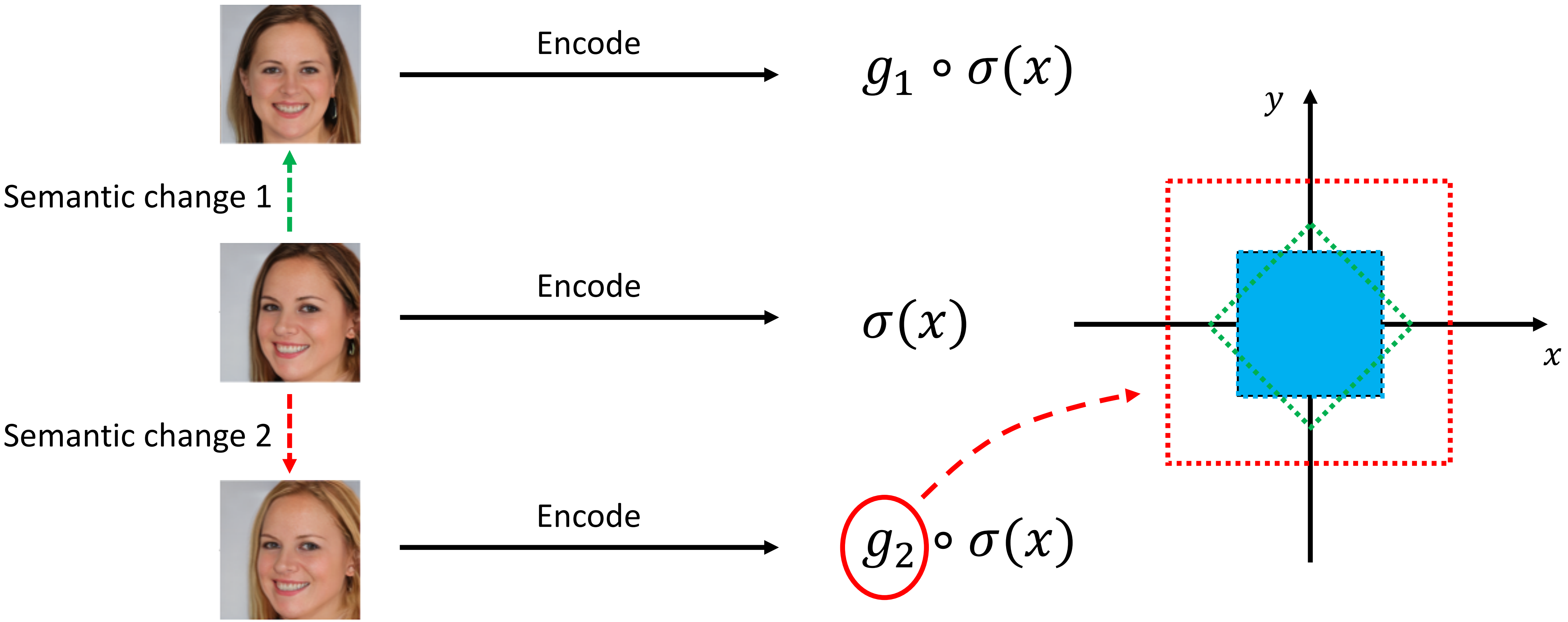
Rethinking Disentanglement Learning



Suppose we encode an image as a state of the latent space.

The different image changes now correspond to different latent transformations.

Rethinking Disentanglement Learning



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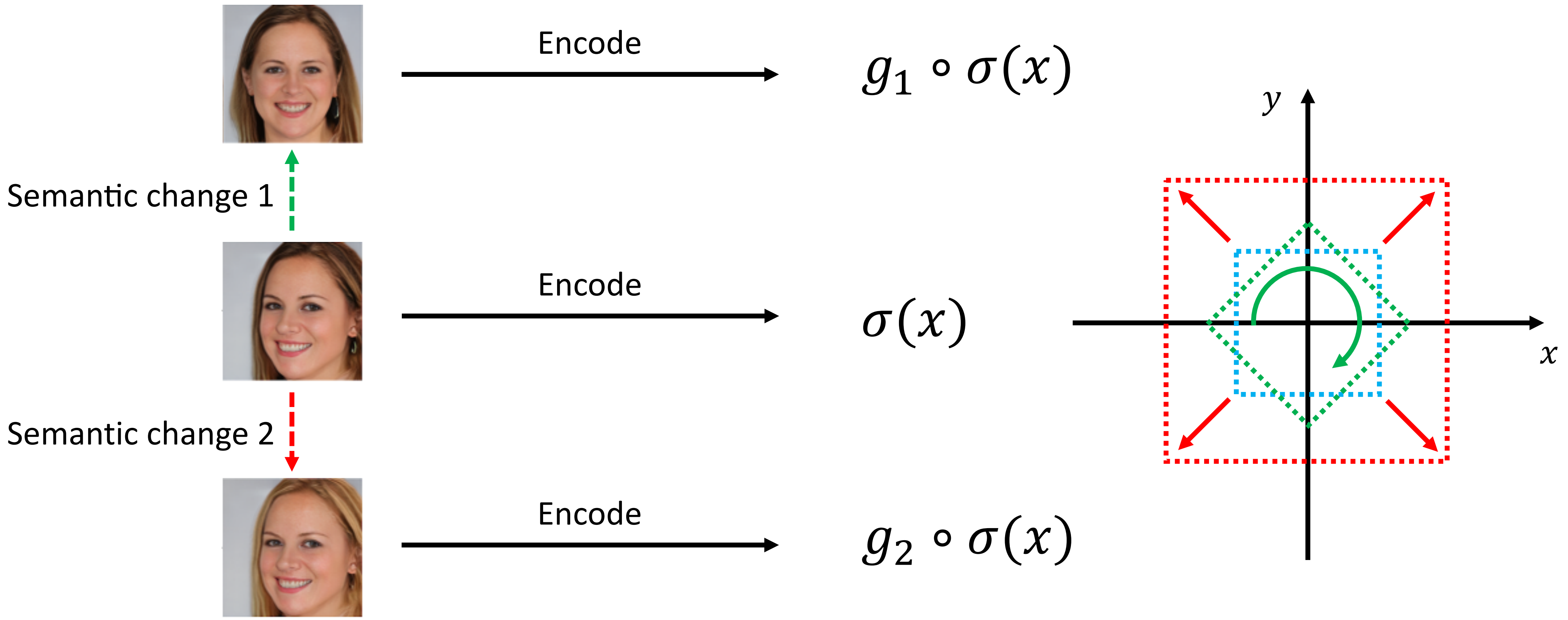
Rethinking Disentanglement Learning



Semantic 1 $= g_1$: rotating transformation.

Semantic 2 $= g_2$: scaling transformation.

Rethinking Disentanglement Learning

A brief comparison:

Old fashion:

Semantic 1 $= g_1$: dim-1 translation.

Semantic 2 $= g_2$: dim-2 translation.

New fashion:

Semantic 1 $= g_1$: rotating transformation.

Semantic 2 $= g_2$: scaling transformation.

OK, but why do we need the new fashion?

Rethinking Disentanglement Learning

A brief comparison:

These translations are fixed and predefined.

Old fashion:

Semantic 1 $= g_1$: dim-1 translation.

Semantic 2 $= g_2$: dim-2 translation.

New fashion:

Semantic 1 $= g_1$: rotating transformation.

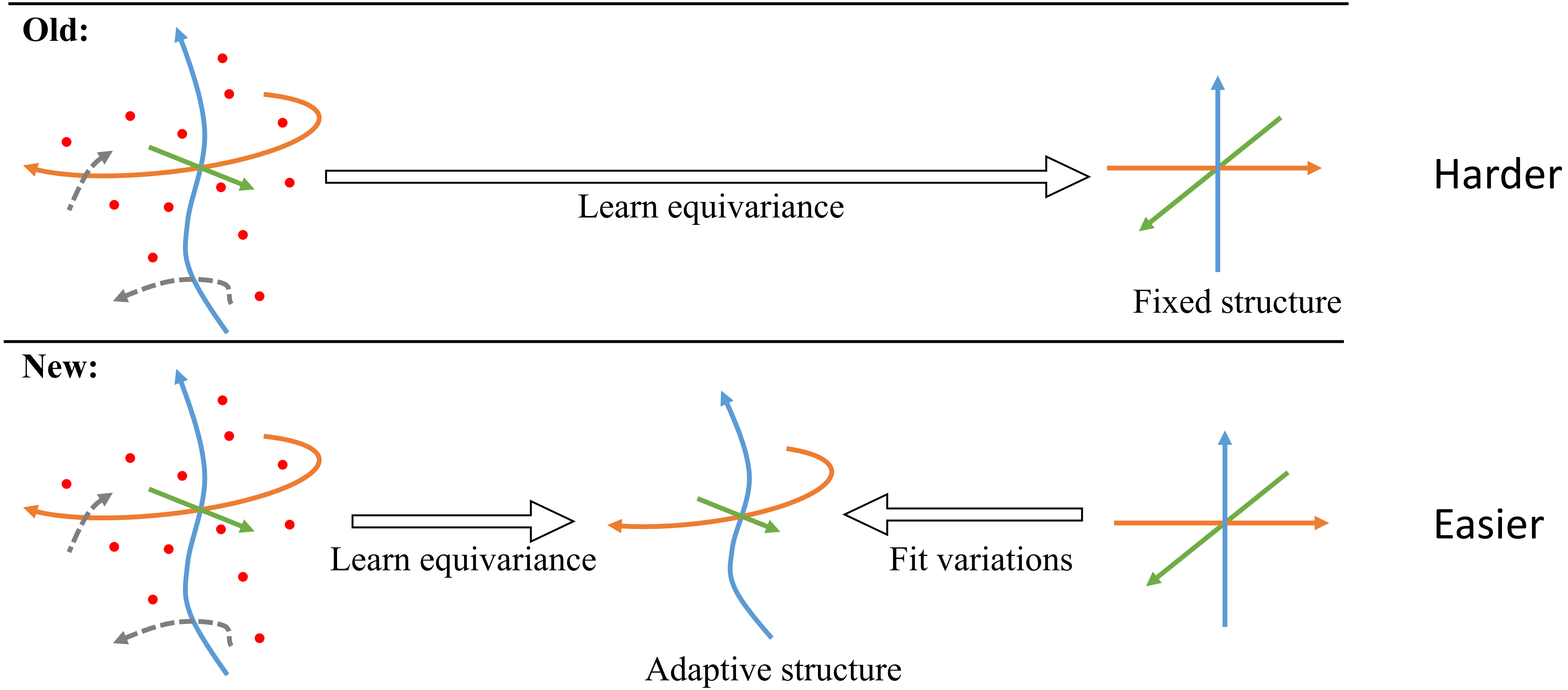
Semantic 2 $= g_2$: scaling transformation.

These transformations can be learned adaptively!

Rethinking Disentanglement Learning

Hypothesis:

The target equivariance can be more easily learned if an adaptive transformation structure is used to capture the semantic variations than a fixed structure.



In this work, we use a learnable group structure to achieve this goal.

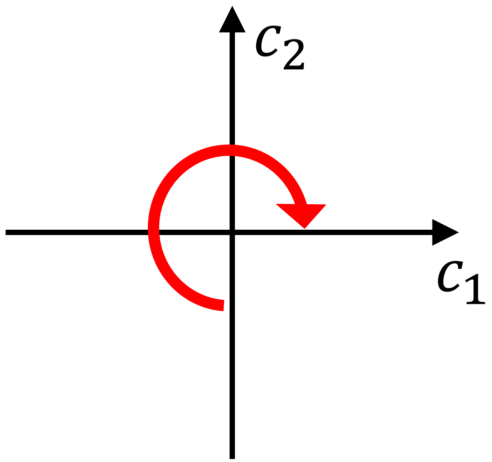
And in this work, we only consider continuous matrix groups.

An Impression on Group

A group element

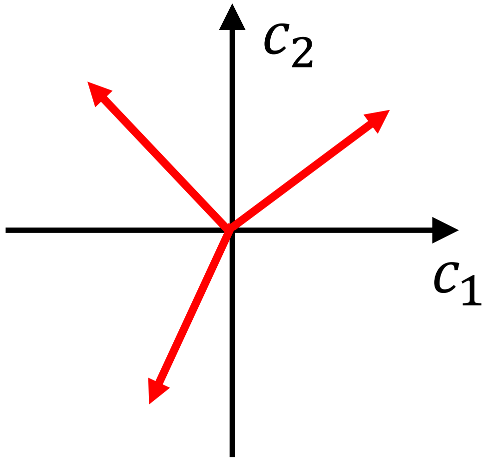
$$\sigma'(x) = g \circ \sigma(x)$$

$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$



Or

$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} S_\theta & 0 \\ 0 & S_\theta \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$



The group element $g \in G$ maps a vector space to itself: $\mathbb{R}^2 \to \mathbb{R}^2$.

An Impression on Group

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$$\begin{bmatrix} \cos\phi' & \sin\phi' \\ -\sin\phi' & \cos\phi' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

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$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} S_\theta & 0 \\ 0 & S_\theta \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

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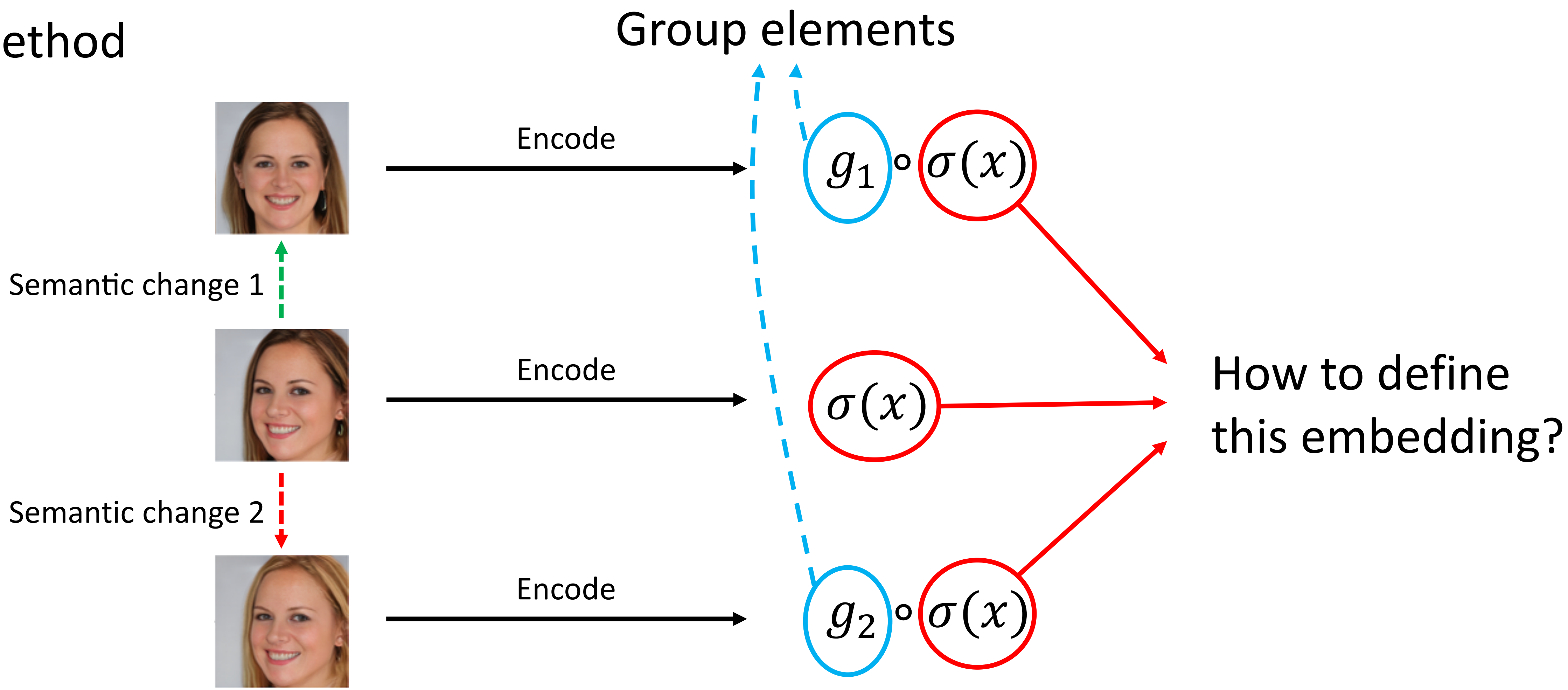
Now the group element $g \in G$ maps a group structure to itself: $G \to G$.

Method



We assume there is a canonical sample x_0, and every other sample is transformed from the canonical one:

$$\sigma(x) = g_{0\to x} \circ \sigma(x_0),$$

and:

$$g_1 \circ \sigma(x) = (g_1 \circ g_{0\to x}) \circ \sigma(x_0);$$

$$g_2 \circ \sigma(x) = (g_2 \circ g_{0\to x}) \circ \sigma(x_0).$$

Method: (1) Group representation

We assume there is a canonical sample x_0, and every other sample is transformed from the canonical one:

$$\sigma(x) = g_{0\to x} \circ \boxed{\sigma(x_0)}, \quad \text{and:} \quad \begin{aligned} g_1 \circ \sigma(x) &= (g_1 \circ g_{0\to x}) \circ \boxed{\sigma(x_0)}; \\ g_2 \circ \sigma(x) &= (g_2 \circ g_{0\to x}) \circ \boxed{\sigma(x_0)}. \end{aligned}$$

It's the group $g \in G$ that defines the representation structure (relation between samples).

The canonical embedding $\sigma(x_0)$ can be seen as a constant.

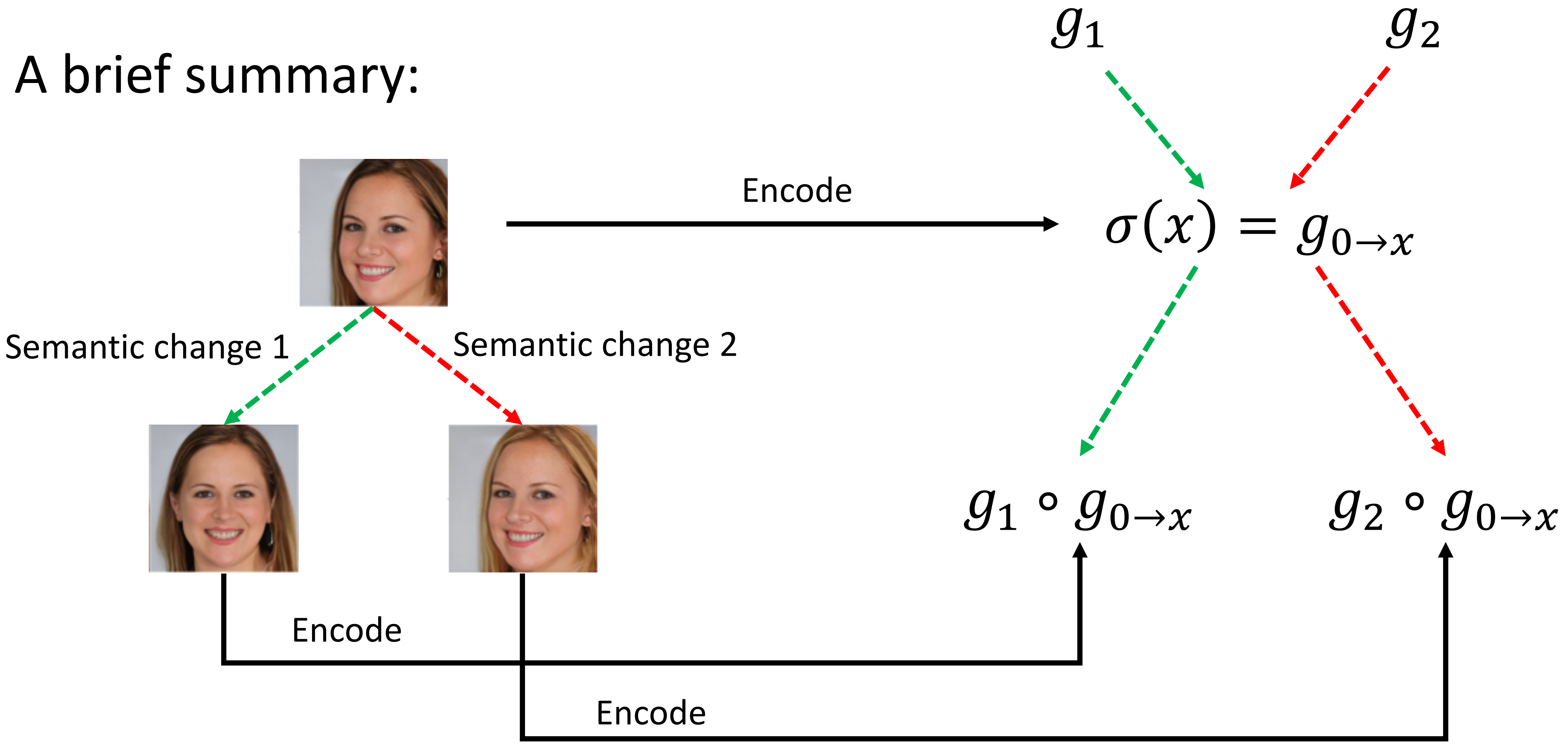
We propose to set $\sigma(x_0)$ to be a fixed value: the group identity element (e).

$$\sigma(x) = g_{0\to x} \circ \sigma(x_0) = g_{0\to x} \circ e = g_{0\to x},$$

Now the samples are embedded on a group structure.
We name this embedding as the 'group representation'.

Method: (1) Group representation

A brief summary:



…looks cool, but how to disentangle $\sigma(x) = g_{0\to x}$?

Method: (1) Group representation

Expectation:

$$\sigma(x) = g_{0\to x} = \underbrace{g_1(x)}_{\text{Semantic 1}} \circ \underbrace{g_2(x)}_{\text{Semantic 2}} \ldots \circ \underbrace{g_m(x)}_{\text{Semantic m}} \circ \underbrace{e}_{\text{Canonical}}$$

We expect every embedding to be decomposed into subgroup actions.

When a new semantic change comes:

$$g_2 \circ \sigma(x) = g_2 \circ g_{0\to x} = g_1(x) \circ (g_2 \circ g_2(x)) \ldots \circ g_m(x) \circ e$$

Method: (1) Group representation

Expectation:

$$\sigma(x) = g_{0\to x} = \underbrace{g_1(x)}_{\text{Semantic 1}} \circ \underbrace{g_2(x)}_{\text{Semantic 2}} \ldots \circ \underbrace{g_m(x)}_{\text{Semantic m}} \circ \underbrace{e}_{\text{Canonical}}$$

In this case, disentanglement on the group representation is achieved via subgroup decomposition.

…looks cool, but how to learn this decomposable group representation?

Method: (2) Lie Algebra Parameterization

$$\sigma(x) = g_{0\rightarrow x}$$

This representation is on a group structure.

At lease we need a way to obtain elements on a group!

In this work, we focus on Lie group and adopt Lie algebra parameterization:

$$g(t) = \exp\big(A(t)\big), g \in G, A \in \mathfrak{g},$$

$$A(t) = t_1 A_1 + t_2 A_2 + \cdots + t_m A_m, \forall t_i \in \mathbb{R}, A_i \in \mathbb{R}^{d\times d}.$$

Basis

Coordinates

Method: (2) Lie Algebra Parameterization

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Now, a group G is defined through a vector space A.

Method: (2) Lie Algebra Parameterization

Since the Lie algebra is a vector space, we can now use general optimization methods (e.g. SGD, Adam) to learn a group structure!

We view the basis $\{A_i\}_{i=1}^{m}$ as learnable weights in a deep model as it determines the structure of the group.

Method: (2) Lie Algebra Parameterization

In this work, we enforce a straightforward group decomposition, namely 'one-parameter subgroup decomposition':

$$
\begin{aligned}
& g(t) \\
& = \exp(t_1 A_1 + t_2 A_2 + \cdots + t_m A_m) \\
& = \exp(t_1 A_1) \exp(t_2 A_2) \ldots \exp(t_m A_m)
\end{aligned}
$$

$g_1(t_1)$	$g_2(t_2)$	$g_m(t_m)$
Semantic 1	Semantic 2	Semantic m

Unfortunately, this decomposition doesn't hold in general.

Method: (3) Disentangle via Decomposition

A proposition is proposed to enforce this decomposition:

Proposition 1. *If* $A_i A_j = A_j A_i, \forall i, j,$ *then*

$$\begin{aligned}
&\exp(t_1 A_1 + t_2 A_2 + \ldots t_m A_m) \\
&= \exp(t_1 A_1)\exp(t_2 A_2)\ldots\exp(t_m A_m) \\
&= \prod_{\mathrm{perm}(i)} \exp(t_i A_i).
\end{aligned}$$

Proof. See Appendix 1.

We can see this group decomposition is commutative, and it is where the 'commutative' in the title comes from.

Method: (3) Disentangle via Decomposition

Furthermore, we also consider another disentanglement constraint called Hessian Penalty (Peebles et al., 2020) on the group structure:

Proposition 2. *If* $A_i A_j = 0, \forall i \neq j$, *then*

$$H_{ij} = \frac{\partial^2 g(t)}{\partial t_i \partial t_j} = 0,$$

where g is the map defined in Eq. 4.

Proof. See Appendix 2.

This is a stronger constraint than Prop. 1 since $A_i A_j = A_j A_i$ is implied by $A_i A_j = 0$.

Method: (4) Constructing a VAE Model

Before imposing the proposed disentanglement constraints, we first introduce a VAE variant called bottleneck-VAE:

Proposition 3. *Suppose two latent variables z and t are used to model the log-likelihood of data x, then we have:*

$$\begin{aligned} \log p(x) &\geq \mathcal{L}_{bottleneck}(x, z, t) \\ &= \mathbb{E}_{q(z|x)}\mathbb{E}_{q(t|x,z)}\log p(x, z|t) \\ &\quad - \mathbb{E}_{q(z|x)}KL(q(t|x,z)||p(t)) - \mathbb{E}_{q(z|x)}\log q(z|x) \quad (9) \\ &= \mathbb{E}_{q(z|x)q(t|z)}\log p(x|z)p(z|t) \\ &\quad - \mathbb{E}_{q(z|x)}KL(q(t|z)||p(t)) - \mathbb{E}_{q(z|x)}\log q(z|x), \quad (10) \end{aligned}$$

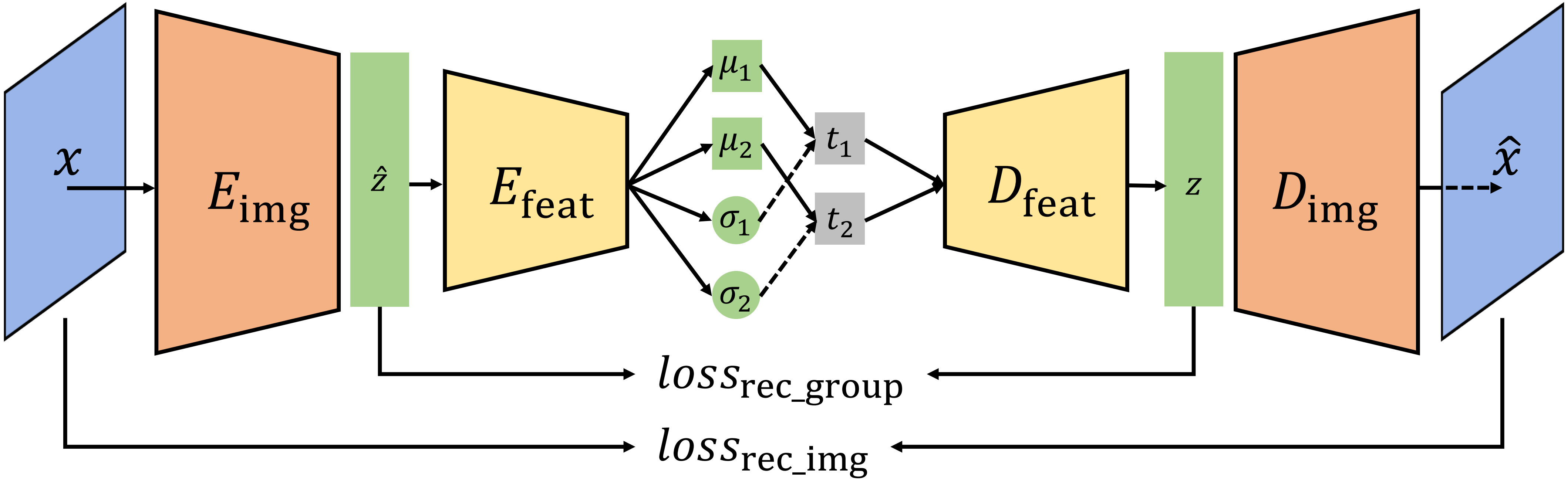
where Eq. 10 holds because we assume Markov property: $q(t|z) = q(t|x, z)$, $p(x|z, t) = p(x|z)$.

Proof. See Appendix 4. □

Method: (4) Constructing a VAE Model

Prop. 3 defines a VAE variant which shares a layer of feature in the encoder and the decoder:

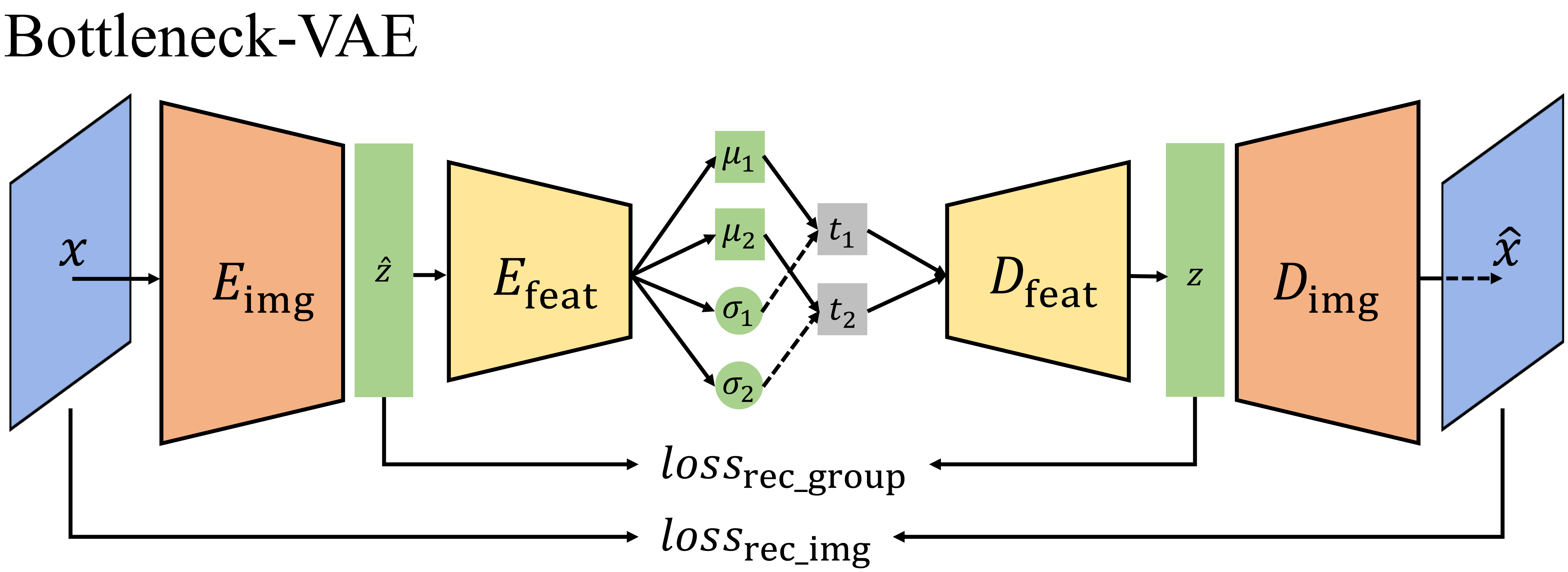
Bottleneck-VAE



In addition to a standard VAE, this model enforce z and $\hat{z}$ to be equal.

Method: (4) Constructing a VAE Model

Our proposed Lie Group VAE is a slight variant of bottleneck-VAE, which has a special group feature decoder:



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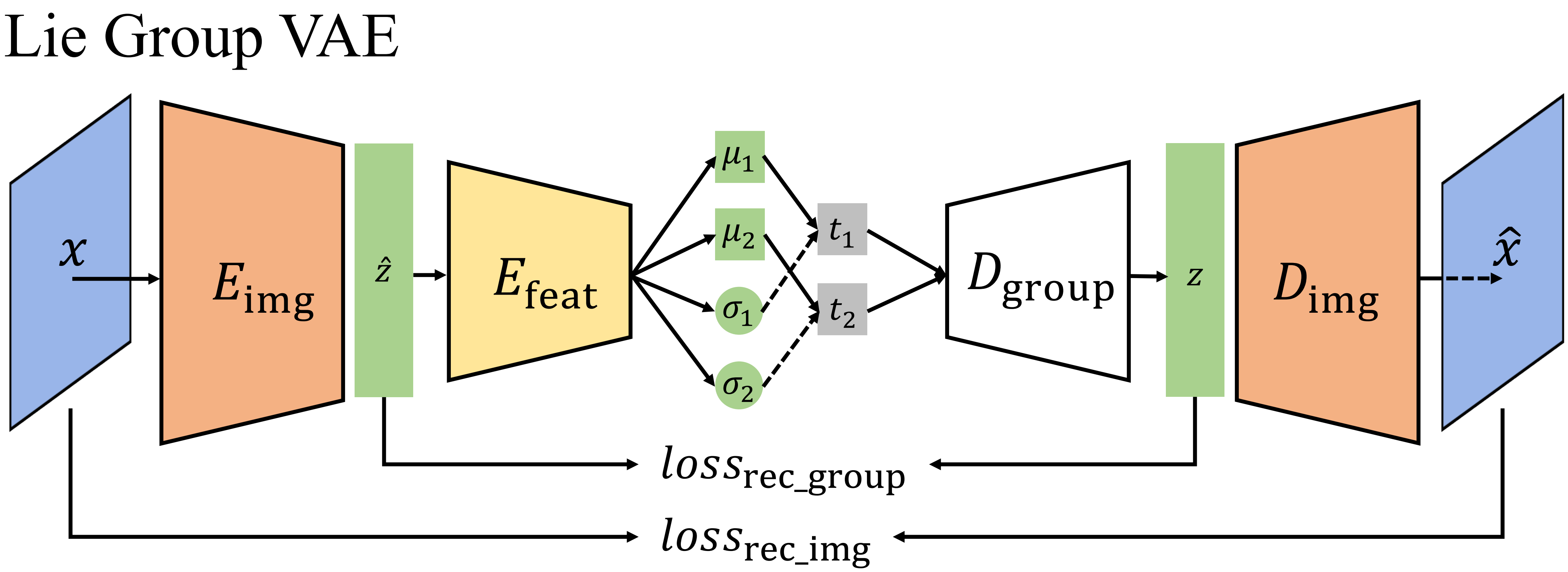
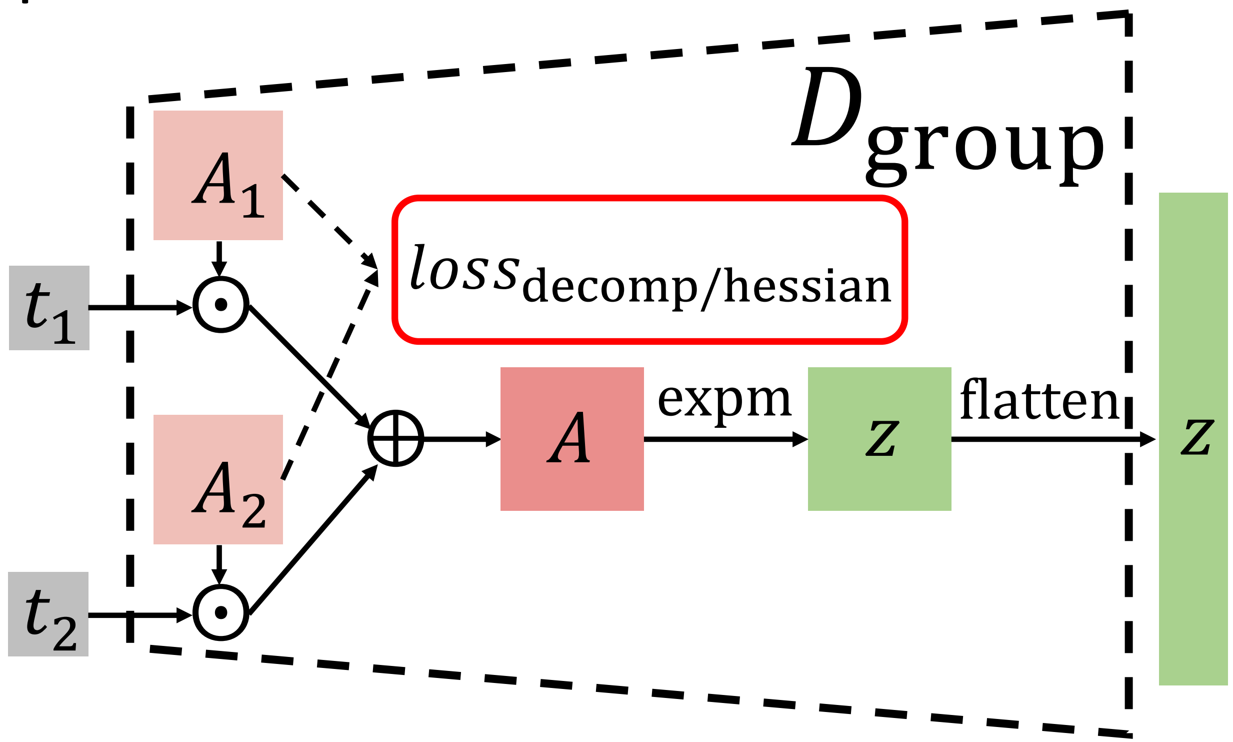


Method: (4) Constructing a VAE Model

Inside the group decoder:



Note that the Lie algebra basis $\{A_i\}_{i=1}^m$ are learnable.

Prop. 1 and Prop. 2 are implemented as regularizations on $\{A_i\}_{i=1}^m$.
We refer a model with these constraints as Commutative Lie Group VAE.

Experiments

Ablation study on DSprites:

Models	FVM	SAP	MIG	DCI
VAE	$69.4_{\pm 10.9}$	$19.7_{\pm 10.6}$	$7.8_{\pm 6.4}$	$8.1_{\pm 4.1}$
+bottle	$74.6_{\pm 8.1}$	$29.2_{\pm 12.1}$	$12.9_{\pm 6.6}$	$11.6_{\pm 3.3}$
+exp	$\mathbf{83.6}_{\pm 3.2}$	$\mathbf{40.7}_{\pm 12.2}$	$\mathbf{17.2}_{\pm 6.8}$	$\mathbf{15.1}_{\pm 2.4}$

Table 1. Ablation study of bottleneck-VAE and exponential map on DSprites.

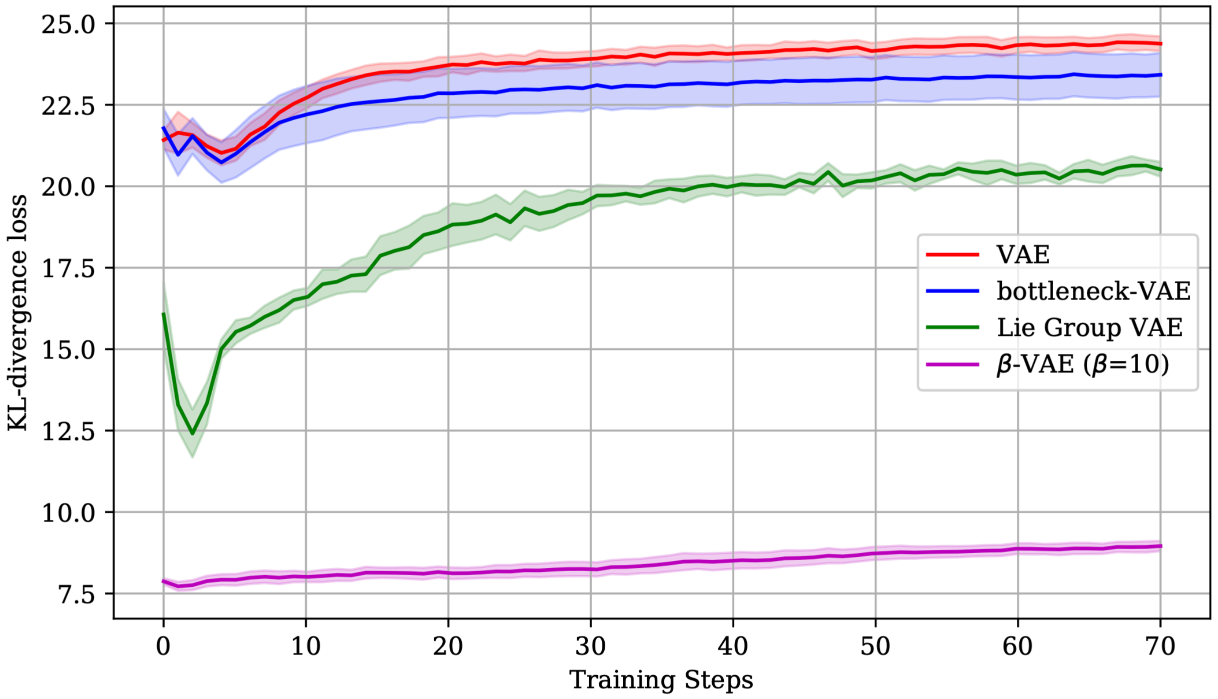


Figure 3. How the KL-divergence loss $(KL(q(t|x)||p(t)))$ evolves during training for different models.

Experiments

Ablation study on DSprites:

λ_{decomp}	FVM	SAP	MIG	DCI
0	$83.6_{\pm 3.2}$	$40.7_{\pm 12.2}$	$17.2_{\pm 6.8}$	$15.1_{\pm 2.4}$
5	$84.0_{\pm 3.9}$	$45.4_{\pm 11.5}$	$20.5_{\pm 6.9}$	$16.8_{\pm 4.3}$
20	$\mathbf{85.8}_{\pm 6.9}$	$48.7_{\pm 8.4}$	$23.6_{\pm 5.0}$	$18.2_{\pm 3.0}$
40	$85.5_{\pm 2.2}$	$\mathbf{50.8}_{\pm 5.0}$	$\mathbf{25.4}_{\pm 6.1}$	$\mathbf{19.7}_{\pm 4.6}$
80	$85.5_{\pm 4.8}$	$47.1_{\pm 8.6}$	$23.3_{\pm 6.2}$	$18.3_{\pm 6.5}$

Table 3. Ablation study of one-parameter decomposition on DSprites.

$\text{Size}_{\text{group}}$	FVM	SAP	MIG	DCI
4	$23.6_{\pm 3.3}$	$6.3_{\pm 6.0}$	$4.2_{\pm 3.9}$	$3_{\pm 0.5}$
9	$57.4_{\pm 5.8}$	$34.1_{\pm 12.9}$	$17.3_{\pm 7.4}$	$12.4_{\pm 4.4}$
25	$79.8_{\pm 2.8}$	$39.6_{\pm 13.4}$	$20.6_{\pm 8.5}$	$19.9_{\pm 3.8}$
64	$82.7_{\pm 3.7}$	$42.2_{\pm 12.5}$	$22.1_{\pm 10.1}$	$\mathbf{20.0}_{\pm 6.8}$
81	$84.4_{\pm 2.6}$	$45.2_{\pm 10.5}$	$23.0_{\pm 8.4}$	$19.6_{\pm 6.3}$
100	$\mathbf{85.5}_{\pm 2.2}$	$\mathbf{50.8}_{\pm 5.0}$	$\mathbf{25.4}_{\pm 6.1}$	$19.7_{\pm 4.6}$

Table 2. Ablation study of group size on DSprites.

λ_{hessian}	FVM	SAP	MIG	DCI
0	$83.6_{\pm 3.2}$	$40.7_{\pm 12.2}$	$17.2_{\pm 6.8}$	$15.1_{\pm 2.4}$
5	$83.8_{\pm 2.4}$	$46.8_{\pm 12.8}$	$19.8_{\pm 8.6}$	$17.5_{\pm 5.6}$
20	$86.1_{\pm 1.8}$	$\mathbf{54.1}_{\pm 1.2}$	$\mathbf{29.7}_{\pm 3.1}$	$\mathbf{23.4}_{\pm 4.1}$
40	$\mathbf{86.2}_{\pm 1.8}$	$48.2_{\pm 1.9}$	$25.2_{\pm 8.4}$	$19.1_{\pm 4.1}$
80	$85.0_{\pm 1.6}$	$43.6_{\pm 11.3}$	$20.1_{\pm 8.4}$	$17.4_{\pm 4.2}$

Table 4. Ablation study of Hessian penalty on DSprites.

Experiments

State-of-the-art comparison:

Model	DSprites	3DShapes
VAE	$69.4_{\pm 10.9}$	$83.6_{\pm 6.5}$
β-VAE	$74.4_{\pm 7.7}$	91 (Kim & Mnih, 2018)
Cascade-VAE	$81.74_{\pm 2.97}$	-
Factor-VAE	$82.15_{\pm 0.88}$	89 (Kim & Mnih, 2018)
Ours	$\mathbf{86.1}_{\pm 2.0}$	$\mathbf{93.2}_{\pm 4.0}$

Table 5. Unsupervised disentanglement state-of-the-art comparison on DSprites and 3DShapes.

Experiments

Qualitative results:

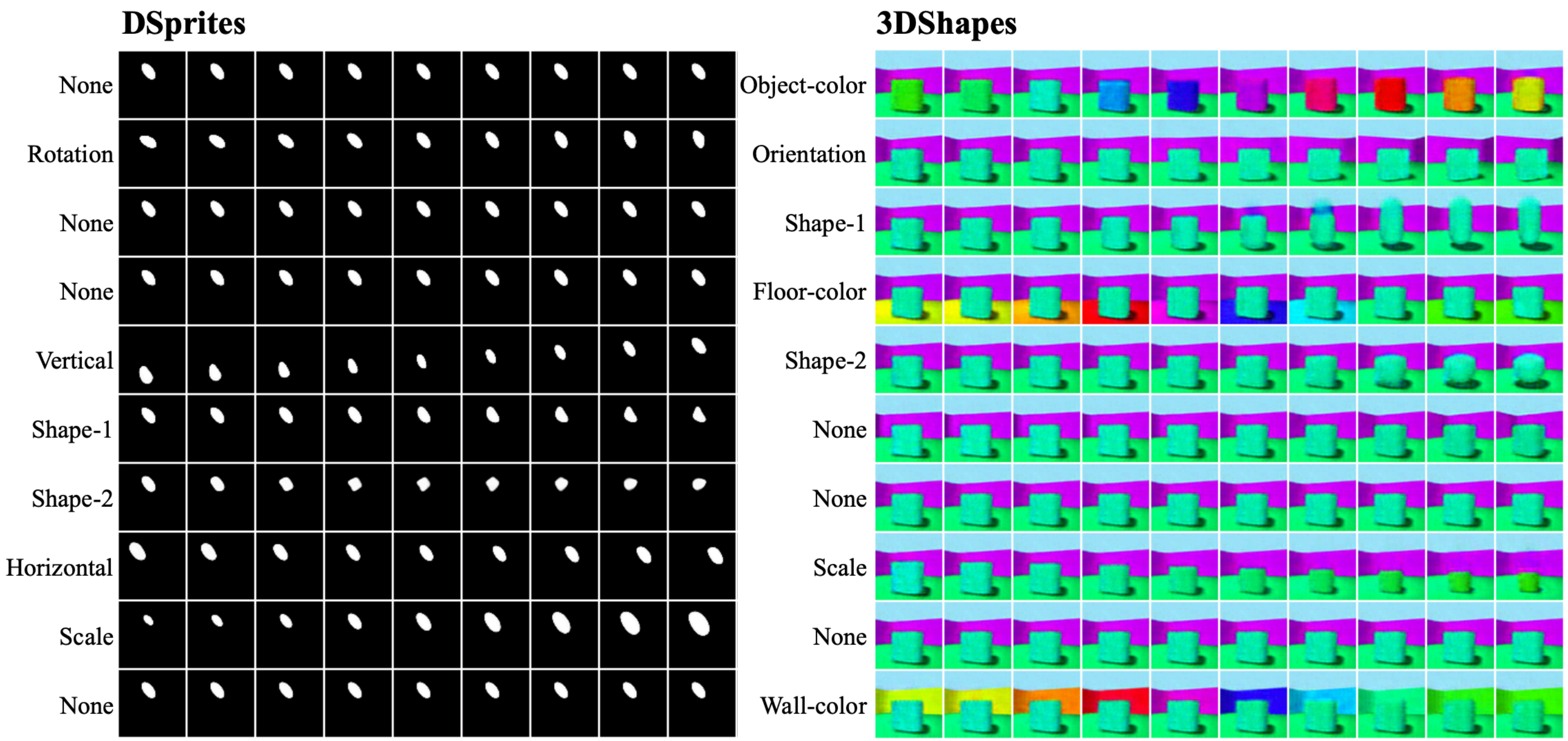
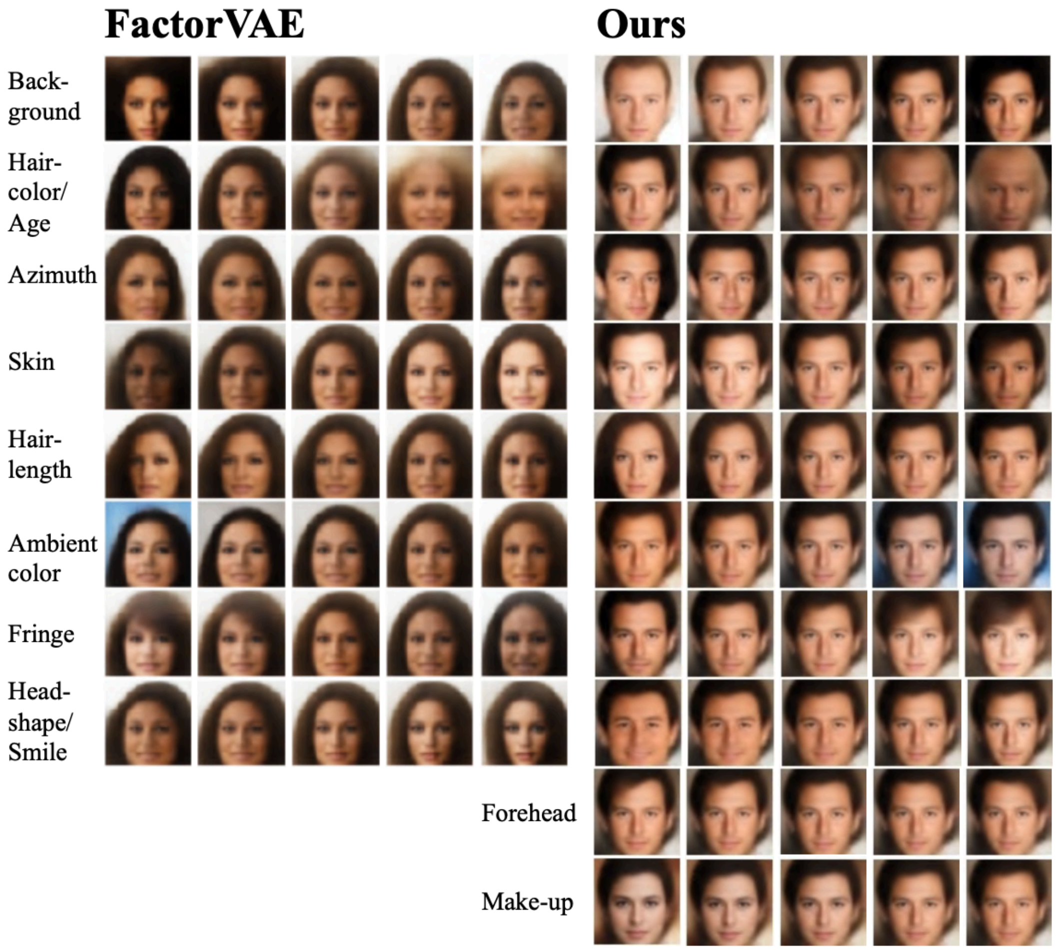
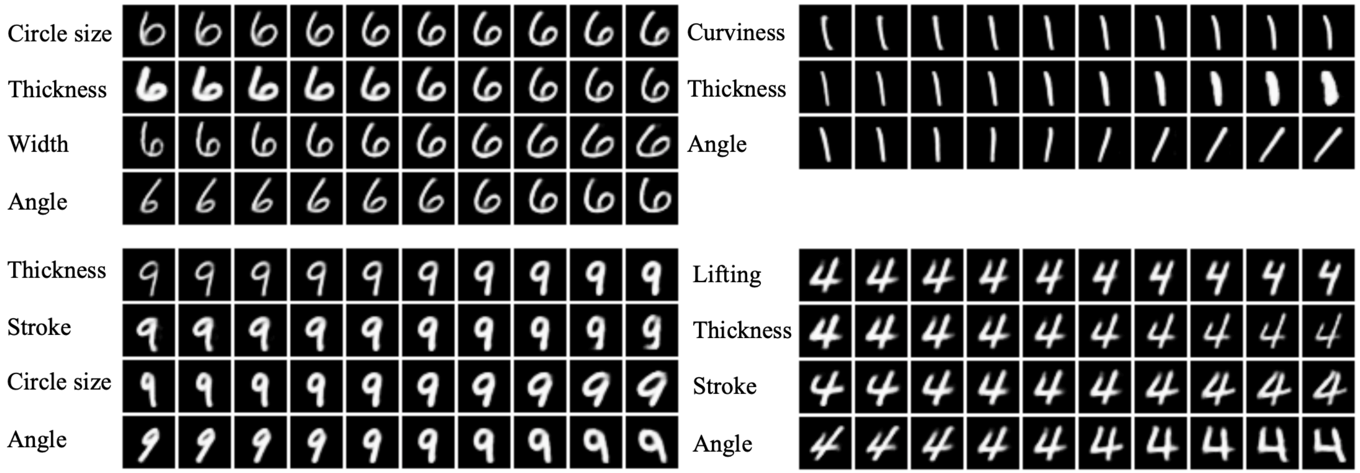


Figure 5. Latent traversals of our Commutative Lie Group VAE on DSprites and 3DShapes datasets.

Experiments

Qualitative results:





Thank you!