Implicit Regularization in Tensor Factorization

Noam Razin* **Asaf Maman*** Nadav Cohen

*Equal contribution

Tel Aviv University

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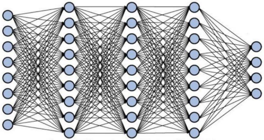
Implicit Regularization in Deep Learning

Neural networks generalize with no explicit regularization even when:

# of learned weights



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# of training examples

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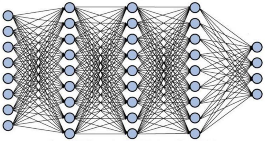
# of learned weights  $\gg$ 

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<u>Conventional Wisdom</u>

GD induces implicit regularization towards low “complexity” predictors

Implicit Regularization in Deep Learning

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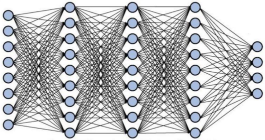

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Goal

Mathematically understand this implicit regularization

Implicit Regularization in Deep Learning

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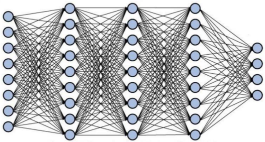

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# of training examples

Conventional Wisdom

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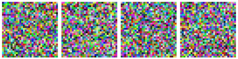
Mathematically understand this implicit regularization

Challenge

Lack complexity measures that capture essence of natural data

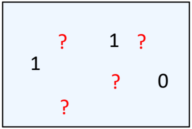
✔ **low complexity**

✘ **high complexity**



Common Testbed: Matrix Factorization (MF)

Matrix completion: recover unknown matrix given subset of entries



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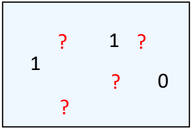

prediction task over 2 input variables

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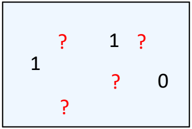

prediction task over 2 input variables

Natural complexity measure: matrix rank

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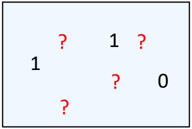

$\Leftrightarrow$ prediction task over 2 input variables

Natural complexity measure: matrix rank

Matrix Factorization

Parameterize solution as product of matrices and fit observations with GD

$$\begin{bmatrix} 4 & ? & ? & 4 \\ ? & 5 & 4 & ? \\ ? & 5 & ? & ? \end{bmatrix} = \boldsymbol{W_L} * \cdots * \boldsymbol{W_2} * \boldsymbol{W_1}$$

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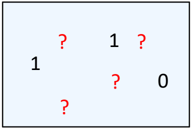

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MF $\longleftrightarrow$ matrix completion via linear NN

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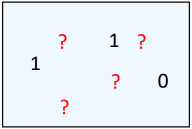

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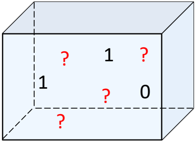
Past Work *(e.g. Arora et al. 2019, Razin & Cohen 2020, Li et al. 2021)*

In MF (with small init and step size) implicit regularization minimizes rank

Beyond Matrix Factorization: Tensor Factorization (TF)

Tensor completion: recover unknown tensor given subset of entries



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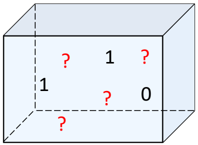

multi-variable prediction task

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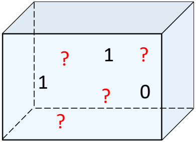

multi-variable prediction task

Tensor Factorization

Parameterize solution as sum of outer products and fit observations via GD

$$\sum_{r=1}^{R} \mathbf{w}_r^1 \otimes \cdots \otimes \mathbf{w}_r^N$$

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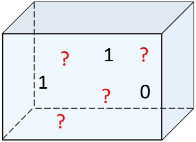

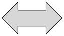

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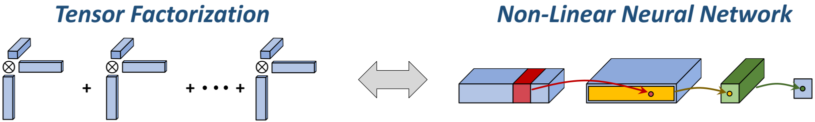

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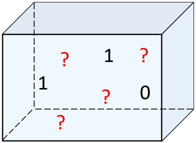

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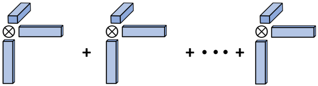
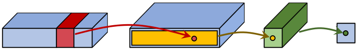


Non-Linear Neural Network



Tensor rank: min # of components (R) required to express a tensor

Dynamical Analysis of Implicit Regularization

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Theorem

In training TF (with small init and step size): $\frac{d}{dt}\|\otimes_{n=1}^{N}\mathbf{w}_r^n\| \propto \|\otimes_{n=1}^{N}\mathbf{w}_r^n\|^{2-\frac{2}{N}}$

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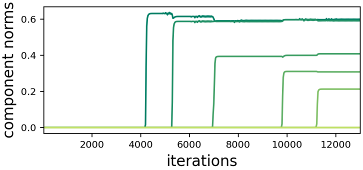
Experiment

Completion of low rank tensor via TF



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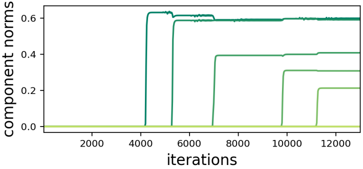

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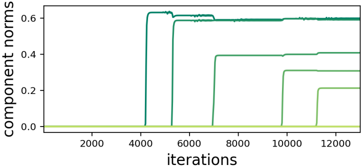

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Proposition (under technical conditions)

If tensor completion has rank 1 solution, then TF will reach it

Tensor Rank as Measure of Complexity

Recall

Goal: understanding implicit regularization in DL

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Tensor rank may shed light on both implicit regularization of NNs and properties of real-world data translating it to generalization